Entropy Jump Across an Inviscid Shock Wave

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Abstract

The shock jump conditions for the Euler equations in their primitive form are derived by using generalized functions. The shock profiles for specific volume, speed, and pressure are shown to be the same, however density has a different shock profile. Careful study of the equations that govern the entropy shows that the inviscid entropy profile has a local maximum within the shock layer. We demonstrate that because of this phenomenon, the entropy propagation equation cannot be used as a conservation law.

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1 Introduction

A consequence of the nonlinearity of the equations of motion is the steeping of compression waves into a shock wave. Within the shock layer, the gradients of velocity and temperature become large, and irreversible thermodynamic processes caused by friction and heat conduction become dominant. At high Reynolds numbers, the shock-layer thickness is of the order of several mean free paths; for all practical purposes, the shock layer can be represented as a mathematical abstraction that corresponds to a surface across which the flow variables experience a sudden jump. Away from this discontinuity surface, viscous and heat conduction effects are usually negligible and the inviscid equations of motion model the flow well. Remarkably the information needed to account for the final outcome of the irreversible processes that take place within the shock layer is contained in the inviscid equations.

The study of shock waves is 150 years old. The jump conditions satisfied by the conservation of mass and momentum were discovered by Stokes [5] in the middle of the 19th century. Stokes' excitement at making this discovery is evident in his paper: "These conclusions certainly seem sufficiently startling; yet a still more extraordinary result...the result, however, is so strange..." The shock jump condition associated with the conservation of energy was implicit in an investigation conducted by Rankine [7] in 1870; however a precise exposition was not made until the work of Hugoniot [5] in 1889. The increase in entropy across a shock was a more difficult concept to grasp. The leading fluid dynamicists in England (Stokes, Kelvin, and Rayleigh) questioned the validity of the shock discontinuity because it violated the conservation of entropy. The correct principles were not well understood, and did not appear in their present form until around 1915.

Detailed studies of the viscous shock layer emerged several years later with the work of Becker [1], who solved exactly the one-dimensional equations of a real fluid. In a related study, Morduchow and Libby [6] found the exact entropy distribution across the shock layer of a viscous heat-conducting gas. Morduchow and Libby observed, see Figure 1, that the entropy, unlike the other flow variables that behave monotonically, increases through the shock layer until it reaches a maximum at the center of the layer and then decreases to its expected value on the other side of the shock. Morduchow and Libby explained this phenomenon as follows: "It may at first sight appear that the recovery of mechanical energy on the downstream side of the center of the wave thus indicated by this solution would violate the second law of thermodynamics... However, the second law applies to an entire system–that is, to the end points–and permits energy recovery in separate sections thereof. The negative entropy here might also be interpreted as physical effects that are not taken into account by the governing equations..."

Today, the shock jump conditions are obtained for the inviscid equations by casting them in their integral conservation form. A brief derivation, based on this standard procedure, is given section 2. However, the purpose of this work is to show that the shock jump conditions can be derived from the primitive differential form of the equations. The genesis of the analysis presented here was contained in an unpublished work of Gino Moretti written in the early 1970's. The significance of this work is primarily that it demonstrates how to obtain the shock jump conditions for equations that cannot be cast in a conservation integral. Similar work has been presented by Colombeau in reference 2. An interesting consequence of this exposition is a better understanding of why the entropy equation does not yield the proper jump.

2 Standard Shock Jump Analysis

The derivation of the jump conditions across a shock associated with the Euler equations is well known. See, for example reference 9. The derivation is included here so that it can be contrasted with the "nonstandard" approach introduced in section 3.
The one-dimensional conservation laws for the inviscid flow of a perfect gas are

$$\begin{aligned} &\int \left[\rho_t + (\rho\, u)_x\right]\, dx = 0 \\ &\int \left[(\rho\, u)_t + (\rho\, u^2 + p)_x\right]\, dx = 0 \\ &\int \left[(\rho\, E)_t + (\rho\, u\, H)_x\right]\, dx = 0 \end{aligned} \tag{1}$$

where ρ is the density, p is the pressure, E is the specific total energy, H is the specific total enthalpy such that

$$H = E + \frac{p}{\rho}$$

and u is the velocity of the gas. In the standard analysis for the shock jump conditions, we weaken the usual smoothness requirements associated with the classical notion of a function by introducing the concept of a weak or generalized solution. Basically, the integrand of equation 1 is multiplied by a test function that is at least C^1 smooth and has compact support. Then, we integrate over space and time in the neighborhood of the shock and use integration by parts to move differentiation from the discontinuous fluid variables onto the smooth test function. Thus let equation 1 be symbolically represented by

$$\int (U_t + F_x)\, dx = 0 \tag{2}$$

where

$$U = (\rho, \rho\, u, \rho\, E)^T$$

and

$$F = (\rho\, u, \rho\, u^2 + p, \rho\, u\, H)^T$$

and let the initial conditions be given by

$$U(x, 0) = U_0(x)$$

Let ϕ be a test function that is continuously differentiable and has compact support. Consider the domain D around the shock Σ defined in the rectangle $0 \leq t \leq t_1$ and $a \leq x \leq b$, see figure 2. Let ϕ be zero outside of D and on its boundary. Multiply the integrand of equation 2 by ϕ, integrate over x and t, and use integration by parts to obtain

$$\int_D \int_{t \geq 0} (U\,\phi_t + F\,\phi_x)\,dx\,dt = 0 \tag{3}$$

We say that U is a weak solution of the initial value problem

$$U_t + F_x = 0$$

with initial data U_0 if equation 3 holds for all differentiable test functions ϕ with compact support.

Let D_l be the subset of D on the left of Σ and D_r be the subset on the right of Σ as in figure 2. Assume that U is differentiable everywhere except across Σ; hence, on D_l with the divergence theorem, we find

$$\int_{D_l} \int_{t \geq 0} (U\,\phi_t + F\,\phi_x)\,dx\,dt = \int_{D_l} \int_{t \geq 0} [(U\phi)_t + (F\phi)_x]\;dx\,dt = \int_{\partial D} \phi\,(-U dx + F dt) \tag{4}$$

and similarly for D_r. Because ϕ is zero on the boundary of D, the line integrals are only nonzero along the shock Σ. Let the shock be defined by $x(t)$ and U_l be the value of U on the left of the shock; similarly, let U_r be the value of U on the right side of the shock. Then by using equation 4 and the equivalent expression on the right of Σ, we obtain

$$\int_\Sigma \phi(-[U]dx + [F]dt) = 0$$

where $[U] = U_r - U_l$ and $[F] = F(U_r) - F(U_l)$. Because ϕ is arbitrary,

$$c\,[U] = [F] \tag{5}$$

along Σ, where $c = dx/dt$ is the speed of the shock. Equation (5) results in the following Rankine-Hugoniot (R-H) jump conditions:

$$\begin{aligned} &[\rho(c-u)] = 0 \\ &[\rho\,u(c-u)] + [p] = 0 \\ &[\rho\,E(c-u)] + [p\,u] = 0 \end{aligned} \tag{6}$$

One solution to equation 6 corresponds to no mass flow across the discontinuity and leads to the conditions across a slip line. The other solution results in jumps in pressure, density, and velocity. After some manipulation, the R-H jumps can be expressed as

$$\begin{aligned} &\rho_l\,\tilde{u}_l = \rho_r\,\tilde{u}_r \\ &p_l + (\rho\,\tilde{u}^2)_l = p_r + (\rho\,\tilde{u}^2)_r \\ &\frac{1}{\gamma-1}a_l^2 + \frac{1}{2}\tilde{u}_l^2 = \frac{1}{\gamma-1}a_r^2 + \frac{1}{2}\tilde{u}_r^2 \end{aligned} \tag{7}$$

where $\tilde{u} = u - c$, a is the speed of sound, and γ is the ratio of specific heats. The above relations indicate that

$$\tilde{u}_r \tilde{u}_l = \frac{p_r - p_l}{\rho_r - \rho_l} \tag{8}$$

which is known as Prandtl's relation, and

$$\frac{\tilde{u}_l}{\tilde{u}_r} = \frac{(\gamma+1)p_r + (\gamma-1)p_l}{(\gamma-1)p_r + (\gamma+1)p_l} \tag{9}$$

These results imply that the entropy jumps across a shock. Its jump is given by

$$[S] = \ln \frac{p_r}{\rho_r^\gamma} - \ln \frac{p_l}{\rho_l^\gamma} \tag{10}$$

Although the entropy propagation equation can be expressed in the form of a conservation law as

$$\int [(\rho\, S)_t + (\rho\, u\, S)_x] dx = 0 \tag{11}$$

it cannot be used to obtain the correct entropy jump across a shock wave.

3 Nonstandard Shock Jump Analysis

The shock jump conditions can be derived without relying on the integral conservation laws. The importance of this method is threefold. First, this method provides a means for determining the jump conditions for physical laws that cannot be expressed in conservation form. Second, it will ultimately lead to an understanding of the nature of the entropy structure across an inviscid shock. Thirdly, it may suggest how to derive shock-capturing algorithms with proper jumps from the nonconservation form of the Euler equations.
Consider the following system of equations:

$$\begin{aligned} v_t - v u_x + u v_x &= 0 \qquad (12)\\ u_t + u u_x + v p_x &= 0 \qquad (13)\\ p_t + u p_x + \gamma p u_x &= 0 \qquad (14)\end{aligned}$$

where $v = 1/\rho$. The reason for using v instead of ρ will become clear in section 4.

We look for solutions to v, u, and p of the form

$$\begin{aligned} v &= v_l + [v]\, H(\xi) \qquad (15)\\ u &= u_l + [u]\, K(\xi) \qquad (16)\\ p &= p_l + [p]\, L(\xi) \qquad (17)\end{aligned}$$

where for now we only require that

$$H\,,\, K\,,\, L = \begin{cases} 0 \text{ for } x \to -\infty \\ 1 \text{ for } x \to \infty \end{cases} \tag{18}$$

$\xi = x - ct$, and $[w] = w_r - w_l$. The functions H, K, and L provide a description of the shock profile or structure with end conditions w_l at $x = -\infty$ and w_r at $x = \infty$, where w stands for v, u, and p.
Consider equation 12 and introduce equations 15 and 16:

$$-c[v]H' - (v_l + [v]H)[u]K' + (u_l + [u]K)[v]H' = 0 \tag{19}$$

We can then rewrite equation 19 as

$$\frac{dH}{dK} - \frac{H}{a+K} = \frac{v_l}{[v](a+K)} \tag{20}$$

where

$$a = \frac{u_l - c}{[u]} \tag{21}$$

By integrating equation 20 we obtain

$$H = \frac{v_l}{[v]} + b(a+K) \tag{22}$$

where b is a constant of integration. Now, for $x \to -\infty$, both H and $K \ \to 0$,; therefore, the constant of integration is

$$b = \frac{v_l}{[v]a} \tag{23}$$

and

$$H = \frac{v_l}{[v]}\frac{K}{a} \tag{24}$$

In the same way as for $x \to \infty$, both H and $K \ \to 1$,; therefore,

$$\frac{v_l}{[v]a} = 1 \tag{25}$$

This last relation, together with equation 21, gives (with $\tilde{u} = u - c$)

$$\frac{\tilde{u}_r}{\tilde{u}_l} = \frac{\rho_l}{\rho_r} \tag{26}$$

which is the R-H jump for the conservation of mass equation found in section 2 (equation 6). By using equations 24 and 25, we find that

$$H = K \tag{27}$$

Now consider equation 13; if we introduce equations 15 - 17, then

$$-c[u]K' + (u_l + [u]K)[u]K' + (v_l + [v]H)[p]L' = 0 \tag{28}$$

With equations 27 and 25, we can rewrite equation 28 as

$$\frac{dL}{dK} + \frac{[u]^2}{[v][p]} = 0 \tag{29}$$

We can integrate

$$L + \frac{[u]^2}{[v][p]}K = d \tag{30}$$

where d is a constant of integration. As $K \to 0$, $L \to 0$, we can conclude that $d = 0$. Hence, for $K \to 1$, $L \to 1$,

$$\frac{[u]^2}{[v][p]} = -1 \tag{31}$$

This reduces to Prandtl's relation:

$$\tilde{u}_r \tilde{u}_l = \frac{p_r - p_l}{\rho_r - \rho_l} \tag{32}$$

If we use equation 31 in equation 30, we get

$$K = L \tag{33}$$

From the first two of equations 12 - 14, we find that the functions H, K, and L must be the same to obtain solutions as in equations 15 - 17.
Now consider equation 14 and introduce equations 16 and 17:

$$[p](u_l - c)L' + \gamma p_l[u]K' + [u][p](KL' + \gamma LK') = 0 \tag{34}$$

However, we see from equation 33 that $K = L$ such that

$$[p](u_l - c)K' + \gamma p_l[u]K' + [u][p](\gamma + 1)KK' = 0 \tag{35}$$

Let us integrate equation 35 from $x = -\infty$ to $x = \infty$ as

$$\{[p](u_l - c) + \gamma p_l[u]\} \int_0^1 dK + [u][p](\gamma + 1) \int_0^1 K dK = 0 \tag{36}$$

which can be reduced to the final R-H jump condition:

$$\frac{\tilde{u}_l}{\tilde{u}_r} = \frac{(\gamma + 1)p_r + (\gamma - 1)p_l}{(\gamma - 1)p_r + (\gamma + 1)p_l} \tag{37}$$

This analysis has shown that if $H = K = L$ (i.e., if the shock profiles for v, u, and p are identical), then the full set of correct R-H jump conditions can be recovered from the equations in primitive form.

4 Multiplication of Discontinuous Solutions

Consider equation 35 rewritten as

$$\{[p](u_l - c) + \gamma p_l[u] + [u][p](\gamma + 1)K\}\, K' = 0 \tag{38}$$

If K belongs to the set of C^1 functions, then the equation above only admits $K =$ constant as a solution. If K is allowed to be a discontinuous solution, then K' cannot be factorized from equation 38. Here, we follow the mathematical construction of generalized functions proposed by Colombeau in reference 2 and Colombeau and Le Roux in reference 3 . The main advantage in using this construction is that most of the operations admissible with smooth functions can be defined for discontinuous functions, including differentiation. (See appendix A.) We restrict our attention to the Heaviside function and its derivative, the Dirac delta function.
The Heaviside function is such that

$$H(x) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x > 0 \end{cases} \tag{39}$$

The Dirac delta function $\delta(x)$ is 0 in $[-\infty, 0[\cup]0, \infty]$ and is such that

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Let $C_0^\infty(\Omega)$ denote the set of all C^∞ functions on Ω with compact support. Given $G_1(x)$, $G_2(x)$, and the test function $\Psi(x) \in C_0^\infty(\Omega)$, if

$$\int_\Omega [G_1(x) - G_2(x)]\, \Psi(x)dx = 0$$

for all $\Psi(x)$, then we say that $G_1(x)$ and $G_2(x)$ are associated and write $G_1(x) \sim G_2(x)$. According to the definition above, $G_1 \sim G_2$ does not imply that $G_3\, G_1 \sim G_3\, G_2$, where G_3 is some other function. Consider, for example, H^n (the nth power of the Heaviside function H). We can show that $H^n \sim H$, but $H^n\, H'$ is not associated with $H\, H'$. In fact, we have

$$H^{n-1}\, H' \sim \frac{1}{n} H' \tag{40}$$

Note that if we replace the associative symbol $\sim$ in equation 40 with the equal sign, then we obtain, for example, $H\, H' = \frac{1}{2} H'$ and multiplication by H yields $H^2\, H = \frac{1}{2} H\, H'$. Now by substituting an equal sign into the equation we get $\frac{1}{3} H' = \frac{1}{2}\frac{1}{2} H'$, which is absurd.

In conclusion, if we replace the equal sign in equation 35 with the symbol $\sim$ then the subsequent integration is fully justified.

In another example, we present a case in which the Heaviside functions describing the shock are not equal because of their behavior at 0. Consider the mass conservation equation

$$\rho_t + \rho u_x + u \rho_x = 0 \tag{41}$$

If we seek a solution for ρ of the form

$$\rho = \rho_l + [\rho] I(\xi) \tag{42}$$

then by using the same procedure above one obtains

$$I = \frac{a+1}{a+K} K \tag{43}$$

where a is given by equation 21. Equation 43 is significant because it shows that the microscopic behavior of the Heaviside function that describes the ρ jump is not the same as that for p or u.

5 Entropy Structure in a Shock Wave

As pointed out in section 2, although a "conservation law" can be written for entropy, this law does not lead to the correct jump. Although this fact is well known, the reason why is not well understood. In this section, we show that the shock profile that corresponds to the entropy cannot be represented by a Heaviside function; hence, the entropy propagation equation does not yield the correct jumps.

Consider equations 28 and 34. Multiply equation 28 by γ and then divide by v; divide equation 34 by p and add the two equations. We obtain the following by simplifying:

$$\gamma(p_l + [p]L)[v]H' + (v_l + [v]H)[p]L' = 0 \tag{44}$$

Because $H = L$ we have $H'L \sim HL' \sim \frac{1}{2}H'$, and we get the R-H jump

$$v_r\ [p_l(\gamma - 1) + p_r(\gamma + 1)] = [v_l\,(p_r(\gamma - 1) + p_l(\gamma + 1)] \tag{45}$$

This shows that the equation

$$\frac{p_t}{p} + \gamma\frac{v_t}{v} + u\left(\frac{p_x}{p} + \gamma\frac{v_x}{v}\right) = 0 \tag{46}$$

has a valid jump. However equation 46 may be rewritten as

$$S_t + u\,S_x = 0 \tag{47}$$

because $dS = dp/p + \gamma\,dv/v$. Now we look for a solution for S of the form

$$S = S_l + [S]\,T(\xi) \tag{48}$$

where T is a Heaviside function. If we substitute equations 48 and 16 into equation 47, then we get

$$\{-c\,T' + (u_l + [u]\,K)\,T'\}\,[S] = 0 \tag{49}$$

either the expression within the braces must be 0 or the jump $[S]$ must be 0. In general, because no relation exists between T and K, the expression within the braces is not 0, hence, we conclude that equation 49 gives the wrong jump, namely $[S] = 0$. In actuality, the problem lies elsewhere. In going from equation 46 to 47, we have gone from an equation with two jumps $[p]$ and $[v]$ to an equation with a single jump $[S]$. By combining the two equations, we have lost some information; furthermore the assumption that the solution can be expressed as in equation 48 is incorrect, which will be shown below.

Consider the following. Without loss of generality, let $v_l = 1$ and $p_l = 1$ and take S_l as the reference state for entropy. Because S by definition is

$$S = \ln p + \gamma \ln v \tag{50}$$

we obtain, given equations 15 and 17 and the fact that $L = H$,

$$\frac{dS}{d\xi} = \frac{dS}{dH} H' = \frac{(1 + [v]H)^{\gamma-1} \{(\gamma+1)[p][v]H + [p] + \gamma[v]\}}{pv^{\gamma}} H' \tag{51}$$

Because the jump $[p]$ is

$$[p] = -\frac{2\gamma[v]}{(\gamma+1)[v] + 2} \tag{52}$$

we can obtain the following by substituting and simplifying:

$$\frac{dS}{d\xi} = \frac{\{1 + [v]H\}^{\gamma-1}}{pv^{\gamma}} (1 - 2H) \frac{\gamma(\gamma+1)[v][v]}{(\gamma+1)[v] + 2} H' \tag{53}$$

This equation shows that $dS/d\xi \sim 0$ because $(1 - 2H)H' \sim 0$ see equation 40; hence, S has a maximum at the origin, which is confirmed by a study of the second derivative. As a result, we conclude that S cannot be described by a simple Heaviside function but rather by the sum of two Heaviside functions

$$S = S_l + (S^* - S_l)T(\xi) + (S_r - S^*)N(\xi) \tag{54}$$

where S^* is the value of S at the maximum. (See figure 3.) The figure shows clear that by using a single equation such as equation 47 for the entropy, we cannot determine the two jumps that actually represent the structure of the entropy at the shock. A more technical proof is given in appendix B.

6 Conclusions

The shock jump conditions for the Euler equations in thier primitive form were derived using generalized functions. It was shown that the structure of the shock profile is not the same for all variables. A study of the entropy propagation equation showed that if the shock structure for the entropy is represesnted by a single Heaviside function, then the wrong entropy jump is obtained. It was then shown that the proper representation of the entropy profile requires two Heaviside functions, but not all the information required to specify this profile can be obtained from the entropy propagation equation.

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Appendix

A Some Definitions About Generalized Functions

Here we briefly discuss generalized functions. For more details on this subject, see references 8 and 4.

A *test function* $\phi(x)$ exists such that

1. $\phi(x)$ is C^∞.
2. $\phi(x)$ has a compact support (i.e., $\phi(x)$ vanishes outside of some compact interval $[a, b]$).

Furthermore, a sequence $\phi_n(x)$ of test functions converges to 0 if

1. for each k, the sequence of the kth derivative $\phi_1^{(k)}(x)$, $\phi_2^{(k)}(x)$,... converges uniformly to 0.
2. every $\phi_n(x)$ vanishes outside a given interval $[a, b]$.

A *generalized function* T is a mapping from the set $\mathcal{D}$ of all test functions into the real or complex numbers such that if $< \cdot, \cdot >$ is the internal product operator, then we have

1. $< T, a\phi(x) + b\psi(x) >= a \ < T, \phi(x) > +b \ < T, \psi(x) >$
2. if $\phi_n(x)$ converges to 0 in the manner defined above, then $< T, \phi_n(x) >$ converges to 0.

Generalized functions are useful because their derivatives are always well defined. In fact, we have the following definition

$$< T', \phi >= - < T, \phi' >$$

B Entropy Structure Proof

Consider equation 50 substitute equations 12 and 14. Because $H = L$, we have

$$S = \ln{(1 + [p]H)} + \gamma \ln{(1 + [v]H)}$$

The equation above can be written as

$$S = \ln{(p_r)}\, F + \gamma\, \ln{(v_r)}\, G$$

where F and G are two Heaviside functions defined as

$$F = \frac{\ln(1+[p]H)}{\ln(p_r)} \qquad\qquad G = \frac{\ln(1+[v]H)}{\ln(v_r)}$$

We show now that $F \sim G$, although they are not identical in the sense that FH' is not associated with GH'. We take a test function $\phi(x)$ (defined in appendix A) and compute the following integrals:

$$\int_{-\infty}^{\infty} F\,\phi(x)\,dx = \int_{-\infty}^{\infty} \frac{\ln(1+[p]H)}{\ln(p_r)}\,\phi(x)\,dx = \int_{0}^{\infty} \phi(x)\,dx$$

$$\int_{-\infty}^{\infty} G\,\phi(x)\,dx = \int_{-\infty}^{\infty} \frac{\ln(1+[v]H)}{\ln(v_r)}\,\phi(x)\,dx = \int_{0}^{\infty} \phi(x)\,dx$$

from which we conclude that $F \sim G$ in accordance with the definition of association. To verify whether $FH' \sim GH'$, we have

$$\int_{-\infty}^{\infty} F\,H' dx = \int_{0}^{1} \frac{\ln(1+[p]H)}{\ln(p_r)}\,dH = 1 + \frac{1}{[p]} - \frac{1}{\ln(p_r)}$$

$$\int_{-\infty}^{\infty} G\,H' dx = \int_{0}^{1} \frac{\ln(1+[v]H)}{\ln(v_r)}\,dH = 1 + \frac{1}{[v]} - \frac{1}{\ln(v_r)}$$

Thus, we conclude that $F\,H'$ and $G\,H'$ are not associated. Therefore, F and G must be considered as two locally different Heaviside functions, although they are associated.

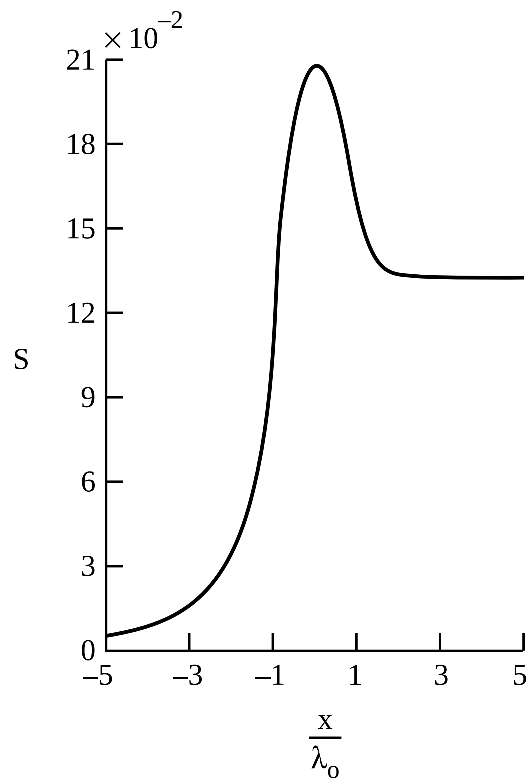

Figure 1: Entropy profiles through viscous shock layer.

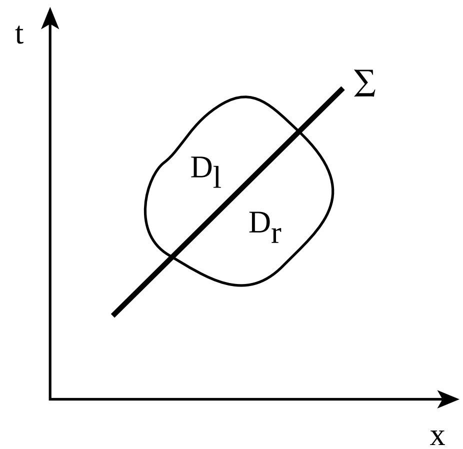

Figure 2: Domain of integration.

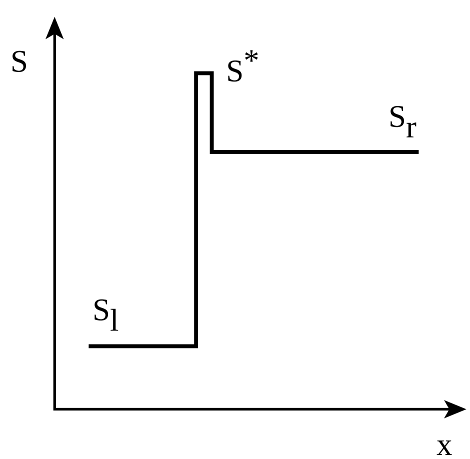

Figure 3: Entropy structure across inviscid shock wave.