

THE INITIAL-VALUE PROBLEM FOR VISCOUS CHANNEL FLOWS ¹

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ABSTRACT

Plane viscous channel flows are perturbed and the ensuing initial-value problems are investigated in detail. Unlike traditional methods where traveling wave normal modes are assumed for solution, this work offers a means whereby completely arbitrary initial input can be specified without having to resort to eigenfunction expansions. The full temporal behavior, including both early time transients and the long time asymptotics, can be determined for any initial disturbance. Effects of three-dimensionality can be assessed. The bases for the analysis are: (a) linearization of the governing equations; (b) Fourier decomposition in the spanwise and streamwise directions of the flow and; (c) direct numerical integration of the resulting partial differential equations. All of the stability data that are known for such flows can be reproduced. Also, the optimal initial conditions can be determined in a straight forward manner and such optimal conditions clearly reflect transient growth data that is easily determined by a rational choice of a basis for the initial conditions. Although there can be significant transient growth for subcritical values of the Reynolds number using this approach, it does not appear possible that arbitrary initial conditions will lead to the exceptionally large transient amplitudes that have been determined by optimization of normal modes. The approach is general and can be applied to other classes of problems where only a finite discrete spectrum exists, such as the boundary layer for example.

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1 INTRODUCTION

The subject dealing with the fate of disturbances in well established flows remains a major topic of fluid mechanics. As the means of obtaining quantitative results for any specific problem has improved, the number of contributions in this area has grown almost without bounds. And, indeed, almost every conceivable prototypical flow has been subjected to scrutiny in this manner. The scheme is well conceived: introduce perturbations into a defined basic flow, linearize the governing equations, and then determine from the initial-value problem the resulting dynamics. In principle this can be done but, in practice, it is a formidable task. Some of the difficulties entail non self-adjoint and singular differential equations among other obstacles. Even the use of the computer and numerical methods have not been easily adaptable. Thus, the net result is that (almost without exception) it was considered adequate if a flow was determined to be stable or unstable. This was done by assuming a separable normal mode solution in the form of traveling waves and then establishing the existence of at least one unstable eigenvalue. No attention was directed to any particular initial-value or the transient period of the dynamics. Indeed, this part of the evolution was not thought to have any significance and, in view of the complications involved in the linear ordinary differential equations, it was left to speculation.

More recently the early transient period for the perturbations has been shown to reveal that a superposition of decaying normal modes may grow initially albeit decay as time goes on. Although the basic origins and recognition of this type of response should properly be given to Kelvin (1887) and Orr (1907a, 1907b), there is an ever widening probe being made; cf. Boberg and Brosa (1988), Gustavsson (1991), Butler and Farrell (1992), Reddy and Henningson (1993), or Trefethen et al. (1993) as major references. Briefly, it has been shown that transient growth can be significant even for subcritical values of the Reynolds number and therefore nonlinearity may ensue, making an exponentially growing normal mode a moot point. In point of fact, this sort of behavior is not hard to grasp because any non self-adjoint differential equation will have eigenvectors that are not orthogonal and therefore there can be algebraic behavior for early time. It is clear that the least damped normal modes should be the ones that are dominant during this early period.

Other approaches for demonstrating algebraic growth have been given by Gustavsson (1979), Benney and Gustavsson (1981), and Criminale and Drazin (1990). In the first case, Laplace transforms were used for the disturbance equations and algebraic behavior was found because there are branch cuts needed for the inversion of the transforms; poles correspond to exponential growth or decay. In the second work, it was shown that, if three-dimensional disturbances are considered, then there can be a resonance between the normal modes of the Orr-Sommerfeld equation and those of the Squire equation. Such resonance does occur and it is for damped exponential modes. The last paper dealt with the existence of the continuous

spectrum as well as the discrete normal modes but, again, it is algebraic temporal behavior that results. The Laplace inversions of Gustavsson that led to algebraic growth were also due to the continuous rather than the discrete spectrum but, unlike the Criminale and Drazin presentation, general initial-values were not taken into account.

The results of the various investigations that have been made for channel flows suggest that the early transient growth can be extremely large (factors of 20 to 1000) at subcritical values of the Reynolds number when considering optimal growth determined from variational techniques. It should be emphasized that not one exponentially growing mode is involved in this process and, although not explicitly stated, the implication is that no matter what may be the initial input, this exceptional growth will be present. Such a premise is subject to scrutiny. Granted variational calculations will assign values to the coefficients of the modes of the expansion in terms of the eigenfunctions, but such a determination may not be the same as for a specific initial-value distribution used to determine the relative values of the coefficients of the eigenmodes. As outlined in Betchov and Criminale (1967) and Drazin and Reid (1981), the use of the eigenfunctions in the initial-value problem requires the adjoint differential equation in order to fix the coefficients. Moreover, it is known Schensted (1960) that the set of such functions is complete for channel flows and consequently there is no conceptual difficulty in the prescription. Lastly, there is no continuous eigenspectrum for viscous channel flows. Thus, transient growth in these flows is due entirely to the non orthogonal eigenvectors of the non self-adjoint differential equations. The optimal formulation is due to Farrell (1988), Butler and Farrell (1992) and has been corroborated by Reddy and Henningson (1993). Up to thirty eigenmodes were employed in these works with relative amplitudes ranging from order one to one thousand. Typical output was a display of the normalized kinetic energy of the perturbations. It is in terms of this quantity that shows explicitly that transient growth leads to the large amplitudes cited before there is eventual decay. The initial distributions used for the optimal values are given in Farrell (1988) and Butler and Farrell (1992). In terms of the initial-value problem this is important information and certainly bears heavily on the possible physical realization. Examination of the initial conditions that produce optimal growth show that the optimal initial conditions for strictly two-dimensional disturbances have discernible structure while those that produce optimal growth of three-dimensional disturbances are rather nondescript. Thus it should be relatively easy to choose initial conditions that approximate the three-dimensional optimal conditions while some care must be taken to choose initial conditions which will approximate the optimal two-dimensional disturbance. This aspect of realization of the optimal conditions is a major point of this study. Butler and Farrell (1994) show that a threshold amplitude exists for optimally configured two-dimensional initial conditions. For amplitudes above the threshold, transient growth leads to a nonlinear evolution to quasi-steady finite amplitude structures, and for amplitudes below the threshold, the decay rate for long time

behaviour is predicted by the slowest-decaying Orr-Sommerfeld mode. The optimization procedures using the eigenfunction expansions in these studies are dependent on the fact that the discrete eigenfunctions form a complete set and therefore are not directly applicable to a problem such as the boundary layer where the complete set was shown by Salwen and Grosch (1981) to be a linear combination of discrete and continuum eigenfunctions. Butler and Farrell (1992) did study the boundary layer optimization problem by using the channel flow solutions to represent the continuous spectrum as a discrete set of modes. The solution procedure chosen here does not present the same difficulties in generalization to a problem with a continuous spectrum since it does not depend on the expansion of an initial condition in terms of its eigenfunctions.

The presentation of Gustavsson (1991) was somewhat different. Here, it was considered sufficient to take only one of the normal modes from the Orr-Sommerfeld equation and combine this with six from the accompanying Squire equation. In effect, this is an initial-value problem but extremely limited. Similarly, a measure of the energy is displayed but with the concentration on the energy gained by the Squire mode. The unknown coefficients here were determined by requiring the Orr-Sommerfeld mode to have an energy of unity at time zero. It was shown that, if there is sufficient obliquity of the waves (nearly 90 degrees), the energy for the Squire mode can transiently increase to almost a factor of 100. In this case, three-dimensionality is crucial but this in no way makes it accessible as a method for realizing large transient growth in an arbitrary initial-value problem. Gustavsson did consider the effects of symmetrical and asymmetrical interaction of the two sets of eigenfunctions. There are decidedly different responses depending upon which choice is made. In particular, a symmetric Orr-Sommerfeld mode interacting with an asymmetrical Squire mode leads to the largest amplitude for the energy. In every case, there is eventual decay after the maximum as time advances for subcritical values of the Reynolds numbers.

Like the boundary layer, viscosity is critical to the perturbations in channel flows. Without a viscous fluid, there is no instability in any of these flows when cast in terms of normal modes. Case (1960) investigated plane Couette flow in an inviscid fluid and, by the use of Laplace transforms, demonstrated that there are no normal modes at all, whether decaying or growing. At the same time he also showed that there are other solutions to this problem if one makes the proper analysis. The additional solutions are those due to a continuous eigenspectrum. Although not covered to this extent, plane Poiseuille flow must follow in like manner. Once the fluid is viscous, then the results are reversed, i.e., there is no continuous spectrum for channel flows but it has been proven that there is an infinite number of normal modes and the set is complete (DiPrima and Habetler, 1969). The boundary layer (see Gustavsson 1979, e.g) possesses both spectra because it is a semi-infinite problem. Parenthetically, however, it is interesting to note that there are only a finite number of normal modes for the boundary layer (see Mack 1976; Grosch and Salwen 1978) with the number depending upon

the value of the Reynolds number. Salwen and Grosch (1981) showed how an arbitrary initial disturbance can be expanded in terms of the complete set of discrete and continuum eigenfunctions, but determining an optimal initial condition would be difficult. It is only for infinite Reynolds number that an infinite (all damped) set of normal modes is possible.

Physically, viscosity causes instability in much the same way that a spring would be unstable if the spring force had a time delay. Mathematically, it is a question of phasing of the perturbations. This is best illustrated by considering the equation for the total kinetic energy of the perturbations. The time rate of change for this energy depends upon two essential terms: dissipation due to viscosity and production manifested by one of the Reynolds stress components interacting with the mean flow to transfer energy and over-come the viscous dissipation. In the case of channel flows this process is possible for Poiseuille but not for Couette flow. And, the boundary layer is also unstable in these terms.

Because of the Squire theorem and the fact that a stability boundary was thought to be sufficient, three-dimensionality per se was neglected because it was recognized that purely two-dimensional disturbances had the largest growth factors. Another often overlooked reason is the fact that the Orr-Sommerfeld equation is only fourth rather than sixth order. This result is fortuitous, makes for ease in the mathematics, but omits important physics. The link to resolution of this problem is the Squire mode equation that is coupled to the behavior of the Orr-Sommerfeld equation so long as there is three dimensionality. It is this pair of governing equations that has formed the bases for the cited studies in transient temporal behavior. Except for the investigation of Gustavsson (1979) that uses the Laplace transform for the boundary layer, the work has relied completely on solutions in terms of normal modes. If the transient dynamics is to be important, then the effects of specific initial conditions must be examined together with any optimum strategy. Certainly, three-dimensionality must be treated in a thorough manner. These aspects of the linear perturbation problem are the central goals of this presentation. This is done with the use of numerical integration of the governing partial differential equations. In no way is an expansion in normal modes suggested but, at the same time, it will be seen that all of the known results of classical stability theory as well as the optimization problem can be reproduced. It is further stressed how important are the details of the perturbation field, most notably the vorticity.

2 BASIC GOVERNING EQUATIONS

For plane viscous channel flows, the fluid is taken as incompressible with the basic flow parallel, $U = U(y), V = W = 0$. Then, the nondimensional linearized equations of motion can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + \frac{dU}{dy} v + \frac{\partial p}{\partial x} = Re^{-1} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right], \quad (2)$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = Re^{-1} \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right], \quad (3)$$

and

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + \frac{\partial p}{\partial z} = Re^{-1} \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right]. \quad (4)$$

Here, $Re = U_o h / \nu$ is the Reynolds number, where h is the channel half-width, U_o the centerline velocity and ν the kinematic viscosity. Time is nondimensionalized by the advective time scale h/U_o . On using the Fourier transformations defined with respect to x and z as

$$\check{v}(\alpha; y; \gamma; t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v(x, y, z, t) e^{i(\alpha x + \gamma z)} dx dz, \quad (5)$$

equations (1) to (4) become

$$-i(\alpha \check{u} + \gamma \check{w}) + \frac{\partial \check{v}}{\partial y} = 0, \quad (6)$$

$$\frac{\partial \check{u}}{\partial t} - i\alpha U \check{u} + U' \check{v} - i\alpha \check{p} = Re^{-1} \left[\frac{\partial^2 \check{u}}{\partial y^2} - \tilde{\alpha}^2 \check{u} \right], \quad (7)$$

$$\frac{\partial \check{v}}{\partial t} - i\alpha U \check{v} + \frac{\partial \check{p}}{\partial y} = Re^{-1} \left[\frac{\partial^2 \check{v}}{\partial y^2} - \tilde{\alpha}^2 \check{v} \right], \quad (8)$$

and

$$\frac{\partial \check{w}}{\partial t} - i\alpha U \check{w} - i\gamma \check{p} = Re^{-1} \left[\frac{\partial^2 \check{w}}{\partial y^2} - \tilde{\alpha}^2 \check{w} \right], \quad (9)$$

respectively, with $U' = dU/dy$ and $\tilde{\alpha}^2 = \alpha^2 + \gamma^2$.

The Squire transformation, written as

$$\alpha \check{u} + \gamma \check{w} = \tilde{\alpha} \tilde{u} \quad (10)$$

$$-\gamma \check{u} + \alpha \check{w} = \tilde{\alpha} \tilde{w} \quad (11)$$

and combined with operations on (6) to (9) enables us to obtain the pair of equations

$$\left[\frac{\partial}{\partial t} - i\alpha U \right] \left(\frac{\partial^2 \check{v}}{\partial y^2} - \tilde{\alpha}^2 \check{v} \right) + i\alpha U'' \check{v} = Re^{-1} \left[\frac{\partial^4 \check{v}}{\partial y^4} - 2\tilde{\alpha}^2 \frac{\partial^2 \check{v}}{\partial y^2} + \tilde{\alpha}^4 \check{v} \right] \quad (12)$$

and

$$\left[\frac{\partial}{\partial t} - i\alpha U \right] \tilde{w} = \sin \phi U' \check{v} + Re^{-1} \left[\frac{\partial^2 \tilde{w}}{\partial y^2} - \tilde{\alpha}^2 \tilde{w} \right] \quad (13)$$

where $\sin \phi = \gamma/\tilde{\alpha}$ and \tilde{w} is proportional to the normal vorticity component ($\tilde{\omega}_y = i\tilde{\alpha}\tilde{w}$). The first term on the right hand side of (13) is called the vortex

tilting term which acts as a forcing term to the normal vorticity component. The vortex tilting term is a product of the mean vorticity in the spanwise direction ($\omega_z = -U'$) and the perturbation strain rate ($\partial v/\partial z$), and for a three dimensional disturbance, gives rise to the increase of the normal vorticity component. It is clear that the solutions of (12) and (13) combined with continuity and the Squire transformation are equivalent to solving (6)-(9). Likewise, \check{p} can be determined from (9). In either case, solutions of the equations are subject to imposed initial conditions and the following appropriate boundary conditions at the channel walls

$$\check{v}(\pm 1, t) = \frac{\partial \check{v}}{\partial y}(\pm 1, t) = \check{w}(\pm 1, t) = 0. \quad (14)$$

For the mean velocity, we shall consider both plane Poiseuille flow

$$U(y) = 1 - y^2$$

and plane Couette flow

$$U(y) = y.$$

To evaluate the other velocity components, the quantities \check{v} and \check{w} are first computed from (12) and (13), respectively. Then the Squire transformation (10)-(11) is inverted to give \check{u} and \check{w}

$$\check{u} = -\frac{i \cos \phi}{\bar{\alpha}} \frac{\partial \check{v}}{\partial y} - \sin \phi \check{w}, \quad (15)$$

$$\check{w} = -\frac{i \sin \phi}{\bar{\alpha}} \frac{\partial \check{v}}{\partial y} + \cos \phi \check{w}. \quad (16)$$

By knowing the velocity components, the vorticity components can be determined in a straightforward manner by appealing to their definitions, yielding

$$\check{\omega}_x = \frac{\partial \check{w}}{\partial y} + i\gamma \check{v}, \quad (17)$$

$$\check{\omega}_y = -i\gamma \check{u} + i\alpha \check{w} \equiv i\bar{\alpha} \check{w}, \quad (18)$$

and

$$\check{\omega}_z = -i\alpha \check{v} - \frac{\partial \check{u}}{\partial y}. \quad (19)$$

Finally, we remark here that, if one seeks solutions to (12) and (13) of the form $e^{-i\omega t}$, then (12) becomes the more familiar Orr-Sommerfeld equation and (13) the Squire equation. Solutions of these equations will yield classical normal modes and normally (i) transient dynamics and (ii) effects of various initial conditions are ignored. At sub-critical Reynolds numbers, where the normal modes are damped, transient behavior may be extremely important. A variety of authors used eigenfunction expansions to examine transient dynamics (e.g., Gustavsson, 1991; Butler and Farrell, 1992; Reddy and Henningson, 1993; as main references). In particular,

since the eigenfunctions form a complete set (DiPrima and Habetler, 1969; Herron, 1980), solutions to (12) and (13) were sought in the form

$$\begin{bmatrix} \ddot{v}(y, t) \\ \ddot{w}(y, t) \end{bmatrix} = \sum_j A_j e^{-i\lambda_j t} \begin{bmatrix} v_j(y) \\ w_j^p(y) \end{bmatrix} + \sum_j B_j e^{-i\mu_j t} \begin{bmatrix} 0 \\ w_j(y) \end{bmatrix}$$

where $\{\lambda_j\}$ and $\{\mu_j\}$ are the eigenvalues of the Orr-Sommerfeld and Squire equations, respectively, with the eigenvalues distinct. As already noted, either (i) variational methods are then used to determine optimal growth; or (ii) a finite combination of eigenfunctions, though extremely limited, are then chosen and their subsequent transient behavior followed. This is not necessarily a weakness because an infinite set is available but only a finite number have been used. An alternative, yet novel, approach is to solve the system (12) and (13) directly by a simple numerical scheme. While this approach does not directly select the optimal initial conditions that provide the optimal algebraic growth for a given set of parameters $(\tilde{\alpha}, \phi, Re)$, it does allow one to follow rather easily the transient dynamics of any given prescribed initial condition and to determine if optimal growth can be approximated by realizable initial conditions. Also, simple maximization methods (see section 4.3) can be easily applied to the numerical solution in order to select the optimal initial condition in a rational manner. Farrell and Moore (1992) also integrated the governing equations for oceanic flows, but again their focus was on determining an optimal initial condition by repeated integration of the perturbation equations and its adjoint and not on the dynamics of specific initial conditions. Here we show that the optimal initial condition can be determined in a straight forward manner which circumvents the necessity of using the adjoint solution, is conceptionally easier to understand, and is easier to implement than the adjoint method or the eigenfunction expansion method. The approach, however, is not necessarily computationally faster. Furthermore, this approach is more robust than employing eigenfunction expansions since it can be applied to other classes of problems where only a finite number of normal modes exist; e.g., boundary layers, free shear layers (see Criminale, et al. 1994 for related work on inviscid flows).

3 NUMERICAL SOLUTIONS

The partial differential equations (12) and (13) were solved numerically by the method of lines. The spatial derivatives were center differenced on a uniform grid within the channel while one-sided differences were used at the walls. The resulting system was then integrated in time by a fourth-order Runge Kutta scheme with all calculations done in double precision. The results were checked for convergence by increasing the number of mesh points, varying between 500 mesh points for low Reynolds numbers to a maximum of 10,000 at larger Reynolds numbers. The table below shows the numerically computed growth rate for plane Poiseuille flow as a function of grid points for $Re = 10,000$, $\tilde{\alpha} = 1$ and $\phi = 0^\circ$:

GRID POINTS	GROWTH RATE
500	0.003726
1000	0.003736
2000	0.003739

The exact value from Orszag (1971) is 0.00373967. The number of grid points were sufficient to resolve the boundary layers near the walls. No effort was made to optimize the number of grid points by employing nonuniform meshes. If this were done, far less grid points would be needed. All calculations presented in this paper represent converged solutions.

Before investigating the effects of various initial conditions and their subsequent transient behavior, it was first instructive to compare for plane Poiseuille flow numerically computed growth rates and eigenfunctions to those of the Orr-Sommerfeld at super-critical Reynolds numbers. Figure 1(A) shows the growth rates obtained from the numerical solution of (12) (shown as circles) and those obtained from the Orr-Sommerfeld equation (shown as the solid curve) for $Re = 10,000$ and $\phi = 0^\circ$. The agreement is excellent. The corresponding real and imaginary parts of the eigenfunctions from both the numerical solution (solid) and the Orr-Sommerfeld solution (dashed) are displayed in figures 1(B) and 1(C), respectively, for the case $\tilde{\alpha} = 1$ and $\phi = 0^\circ$. Note that the two curves essentially lie on top of each other. To demonstrate the strength of the procedure, figure 2 shows similar results but at $Re = 10^6$.

4 PERTURBATION ENERGY

As mentioned above, we are particularly interested in the effects of various initial conditions and their subsequent transient behavior at sub-critical Reynolds numbers. In order to examine the evolution of various initial conditions, the energy density in the $(\tilde{\alpha}, \phi)$ plane as a function of time is computed. The energy density is defined as

$$E(t; \tilde{\alpha}, \phi, Re) = \int_{-1}^1 [|\ddot{u}|^2 + |\ddot{v}|^2 + |\ddot{w}|^2] dy. \quad (20)$$

The total energy of the perturbation can be found by integrating (20) over all $\tilde{\alpha}$ and ϕ . A growth function can be defined in terms of the normalized energy density, namely

$$G(t; \tilde{\alpha}, \phi, Re) = \frac{E(t; \tilde{\alpha}, \phi, Re)}{E(0; \tilde{\alpha}, \phi, Re)} \quad (21)$$

measures the growth in energy at time t for a prescribed initial condition at $t = 0$.

Various initial conditions are used to explore transient behavior at sub-critical Reynolds numbers. For the integration of (12) the initial conditions for \ddot{v} are provided in the table below; $\Omega_0 = \Omega_0(\alpha, \gamma)$.

CASE	$\check{v}(y, 0)$
I	$\Omega_0(1 - y^2)^2$
II	$\frac{\Omega_0}{\beta^2} [\cos \beta - \cos(\beta y)]; \quad \beta = n\pi$
III	$\frac{\Omega_0}{\beta^2} [\cos \beta - \cos(\beta y)] \frac{e^{-y^2/4\lambda}}{\sqrt{4\pi\lambda}}; \quad \beta = n\pi$
IV	$\Omega_0 y(1 - y^2)^2$
V	$y \frac{\Omega_0}{\beta^2} [\cos \beta - \cos(\beta y)] \frac{e^{-y^2/4\lambda}}{\sqrt{4\pi\lambda}}; \quad \beta = n\pi$

Note that the first three cases correspond to symmetric initial conditions while the last two are asymmetric. For the integration of (13) the initial conditions for \tilde{w} are provided in the table below.

CASE	$\tilde{w}(y, 0)$
i	0
ii	$\Omega_1 \cos(\beta_0 y); \quad \beta_0 = (2n - 1)\pi/2$
iii	$\Omega_1 \sin(\beta_1 y); \quad \beta_1 = n\pi$

The second initial condition is symmetric while the last is asymmetric. We remark here that, in choosing these particular initial conditions, no attempt was made to find an optimal initial condition that correspond to a maximum transient growth. Also, particular eigenmodes are not investigated since these were studied previously by Gustavsson (1991). Rather, reasonable initial conditions consisting of polynomials and transcendental functions were constructed and their subsequent transient behavior followed. The above set of initial conditions is general enough that it should be sufficient to test whether the optimal transient growth previously determined is realizable or not.

4.1 TWO-DIMENSIONALITY

In this section results are presented for plane Poiseuille flow with $\tilde{\alpha} = 1.48$, $\phi = 0^\circ$ and a Reynolds number of $Re = 5,000$, and for plane Couette flow with $\tilde{\alpha} = 1.21$, $\phi = 0^\circ$ and $Re = 1,000$. For plane Poiseuille flow, this Reynolds number is sub-critical and thus the growth function G will eventually decay in time to zero. Because plane Couette flow is linearly stable, G will always eventually decay. These values correspond to those found by Butler and Farrell (1992) who showed that these values are the best 2-D optimal using variational techniques in the sense that they give the largest algebraic growth.

The growth function is plotted as a function of time in figure 3 for plane Poiseuille flow and in figure 4 for plane Couette flow. In both figures, the

curves in (A) correspond to all possible combinations of the initial conditions for \check{v} and \check{w} with $\Omega_0 = \Omega_1 = \lambda = n = 1$. Except for some minor algebraic growth in the case of plane Couette flow, the growth function decays. This decrease in the amplitude of the maximum growth is somewhat unexpected since previous work shows that by considering optimal initial conditions that substantial growth can occur even for these two dimensional disturbances. Therefore, this issue was pursued further by considering initial disturbance velocity profiles with more zeroes which, in some sense, corresponds to the higher eigenmodes whose inclusion in the optimization analysis was necessary to achieving the high growth rates. Figures 3(B) and 4(B) are for the initial condition (II,i) with various values of n . For plane Poiseuille flow and with $n = 7$ the maximum value is 12, and for plane Couette flow and with $n = 3$ the maximum value is 4.8. In both cases, moderate transient growth is observed, with the maximum growth being lower than that obtained by Butler and Farrell (1992). It is interesting to note that an “optimal” initial disturbance will be found for both the Couette flow and the Poiseuille flow that is consistent with the optimal conditions determined by Butler and Farrell (section 4.3). For Couette flow, the maximum optimal energy growth for this choice of $\tilde{\alpha}$ and ϕ occurs at $t = 8.7$. Here, we observe that the largest growth is for the initial condition with $n = 3$ and the maximum occurs at time $t = 7.8$. The same can be said of Poiseuille flow. The optimal initial conditions produce a maximum at time $t = 14.1$ and the largest growth here is for $n = 7$ that has a maximum at time $t = 14.4$. It is easy to see how these solutions for different values of n can be combined to produce an optimal solution. This issue is explored further in section 4.3.

4.2 THREE-DIMENSIONALITY

In this section results are presented for plane Poiseuille flow with $\tilde{\alpha} = 2.044$, $\phi = 90^\circ$ and a Reynolds number of $Re = 5,000$, and for plane Couette flow with $\tilde{\alpha} = 1.66$, $\phi = 90^\circ$ and $Re = 1,000$. These values appear in Butler and Farrell (1992) and the choice of $\tilde{\alpha}$ corresponds to the streamwise vortex with largest growth. For plane Poiseuille flow the global optimal coincides with the streamwise vortex but not so for Couette flow. In the latter case, the global optimal was shown to be at $\phi \approx 88^\circ$. Here, we are only interested in presenting two representative cases. In all calculations we set $\Omega_0 = \Omega_1 = \lambda = n = 1$. Changing the sign of Ω_1 produced only very minor changes in the solutions. Numerical experiments were also carried out for various combinations of n and λ . In all cases, the largest value of G are for the cases presented below, namely $n = \lambda = 1$, and therefore only these cases are presented here.

The growth function is plotted as a function of time in figure 5 for plane Poiseuille flow and in figure 6 for plane Couette flow. The curves in (A) correspond to the initial condition (i) for \check{w} , while the curves in (B) correspond to (ii) and the curves in (C) correspond to (iii). Comparing figures 3 and 5 for the case of plane Poiseuille flow, and figures 4 and 6 for the case of plane Couette flow, we see that the transient growth is significantly larger for three

dimensional disturbances than those for two dimensional disturbances. For plane Poiseuille flow, the maxima of the symmetric disturbances (labeled I, II, and III) and the maxima of the asymmetric disturbances (labeled IV and V) in figure 5(A) are within 90% of the global maxima reported by Butler and Farrell (1992). They point out that the presence of streamwise vorticity, while passive to nonlinear dynamics (Gustavsson, 1991), can cause the development of streaks which may themselves be unstable to secondary instabilities or possibly produce transient growth of other types of perturbations. For plane Couette flow, the maxima of the symmetric disturbances (labeled I, II, and III) in figure 6(A) are within 97% of the maxima reported by Butler and Farrell, while the maxima of the asymmetric disturbances (labeled IV and V) are significantly smaller. The significance of this is that any initial condition with \bar{v} velocity symmetric and no initial vorticity will give near optimum results when three-dimensionality is considered. This easily explains the growth observed by Gustavsson (1991) when a limited normal mode initial condition was chosen.

Since this large transient growth for three dimensional disturbances is a direct result of the growth of vorticity, it is necessary to ask whether the growth is a special case because the initial vorticity is neglected, or will the growth in energy remain once the energy of non-zero initial vorticity is included in the calculation. Note in figures 5 and 6 that the symmetric disturbances all behave in a similar manner when the initial condition for \bar{w} is given by (i), but differ substantially when conditions (ii) or (iii) are used. It is clear that the substantial growth in energies shown in figure 5(A) is directly attributable to the generation of normal vorticity through the coupling term in equation (13). This produces growth factors of 3900-4200 for symmetric initial profiles (I, II, and III) and growth factors of 2200-2400 for anti-symmetric initial profiles (IV and V). The responses change significantly when the energy of the initial profiles (ii) and (iii) are included in the initial normalization of the growth factor G . The responses to initial profiles I and IV are lowered but still show significant growth. The initial energy of II and IV are larger than those of (ii) and (iii). However, when the energies of the initial normal vorticities (ii) and (iii) are significantly larger or comparable to the energies of the initial velocity profiles II, III, and V, the substantial transient growth as measured by the total energy can decrease by a factor of ten. This is not to say that the transient growth shown in (A) is not also present in (B) and (C), but that the normalization in (B) and (C) reflects a more proper measure of growth that includes the total energy of an arbitrarily chosen initial disturbance and not just an optimal one. This suggests that the algebraic growth as measured here and in previous work is extremely sensitive to the presence of any initial normal vorticity, specifically to the inclusion of the energy associated with the initial streamwise vorticity in normalization of the growth factor. Vorticity components are presented in figure 7 for plane Poiseuille flow with initial conditions given by (II,i) and (II,ii), respectively, showing that the maxima in time of the normal and spanwise vorticity components decrease when the normal vorticity

component is initially non-zero. The streamwise component decays and is unaffected by this change of initial conditions.

Table 1: Maximum values at various Re corresponding to the initial condition (I,i) for plane Poiseuille flow with $\tilde{\alpha} = 2.044$ and $\phi = 90^0$.

Re	t^*	G^*	t^*/Re	G^*/Re^2
500	41.0	45.6	0.0820	1.824×10^{-4}
1000	82.0	181.9	0.0820	1.819×10^{-4}
2000	164.0	726.9	0.0820	1.817×10^{-4}
4000	327.0	2907.0	0.0818	1.817×10^{-4}
5000	409.0	4542.0	0.0818	1.817×10^{-4}

Table 2: Maximum values at various Re corresponding to the initial condition (I,i) for plane Couette flow with $\tilde{\alpha} = 1.66$ and $\phi = 90^0$.

Re	t^*	G^*	t^*/Re	G^*/Re^2
500	69.0	291.4	0.1380	1.165×10^{-3}
1000	139.0	1165.2	0.1390	1.165×10^{-3}
2000	277.0	4660.7	0.1385	1.165×10^{-3}
4000	554.0	18642.4	0.1385	1.165×10^{-3}
8000	1109.0	74569.6	0.1386	1.165×10^{-3}

Finally, shown in Tables 1 and 2 are the times (t^*) for which the growth function attains its maximum (G^*) at various Reynolds numbers for plane Poiseuille and plane Couette flow, respectively. Note that the time scales as Re while the maximum scales as Re^2 , as previously pointed out by Gustavsson (1991).

4.3 OPTIMAL INITIAL CONDITIONS

A mechanism for rapid transient growth when the initial conditions is expressed as a sum of the eigenfunctions has been explained by Reddy and Henningson (1993). The concept is that a group of eigenfunctions are nearly linearly dependent so that, in order to represent an arbitrary disturbance (say), then it is possible that the coefficients can be quite large. Now, since each one of these nearly linearly dependent eigenfunctions have differing decay rates, the exact cancellations that produce the given initial disturbance might not persist in time and thus significant transient growth can occur. This process can (and is) taken a step further in order to determine the optimal initial condition (still expressed as a sum of the non-orthogonal eigenfunctions) that produces the largest relative energy growth for a certain time period. This process is completed and it does have the feature that the

nearly linearly dependent eigenfunctions are multiplied by coefficients three orders of magnitudes greater than the others. This optimal initial condition produces a growth factor of about 20 for the two-dimensional disturbance in Poiseuille flow. However, this optimal growth is nearly destroyed by not including the first eigenfunction (growth drops to a factor of 6 rather than twenty) which seems to indicate that the prior explanation of (initial) exact cancellations by the nearly linearly dependent eigenfunctions is not the entire mechanism. Butler and Farrell (1992) also calculated optimal initial conditions in terms of a summation of the eigenfunctions (although they put no particular emphasis on the importance of using this approach) and re-iterated the importance of nearly linear dependence of the modes to the transient growth. Butler and Farrell (1992) also explained the transient growth of the optimal initial conditions in terms of the vortex-tilting mechanism and the Reynolds stress mechanism, since these (physical) arguments apply no matter what the solution method; perhaps in the end they should be preferred to those based strictly on the mathematical procedure, especially since owing to the non-orthogonality and near linear dependence of the eigenfunctions. The resulting optimal initial disturbance is difficult to visualize physically for someone who is not privy to the calculations.

Using the method that we have been following, an optimization procedure can be determined without resorting to a variational procedure requiring the numerical determination of the eigenfunctions. Now for these channel flows, the difficulties of calculating the eigenvalues and eigenfunctions are well known. A group of eigenfunctions are nearly linearly dependent and the whole set is not orthogonal. Also, the issue of degeneracies in the parameter space must be considered. Thus, an attempt to write a specific initial condition as an eigenfunction expansion (which also requires the determination of the eigenfunctions to the adjoint equation) runs into difficulties for the uninitiated. In this paper, we show that calculating growth for particular initial conditions does not present great difficulty. Furthermore, a closer inspection of initial condition II suggests that each of these disturbances is in essence a single Fourier mode of an arbitrary initial condition. If one were to consider an arbitrary odd function for the \tilde{u} velocity satisfying the boundary conditions written in terms of a Fourier sine series, then the initial condition in the \tilde{v} velocity is given by II. Thus, if one wished to determine an optimal initial disturbance, a maximization procedure could be applied to an arbitrary linear combination of these modes, all of which are initially orthogonal and linearly independent. Clearly, if one wanted to also include non-zero initial vorticity in such an optimization scheme it would not be difficult to include (and these initial conditions are of course very important when modeling real disturbances as opposed to optimal disturbances). The results presented here show that, if included in the optimization procedure, the initial vorticity modes would not contribute to the optimal solution for the cases considered.

To start our optimization scheme, we consider the total solution $\vec{u} = (\check{u}, \check{v}, \check{w})$ to be the sum

$$\vec{u}(y, t) = \sum_{k=1}^N (a_k + ib_k) \vec{u}_k(y, t)$$

where each of the vectors $\vec{u}_k(y, t)$ represents a solution to equations (12) and (13) subject to the initial conditions

$$\vec{u}(y, 0) = \left\{ \begin{array}{c} \cos \phi \sin k\pi y \\ \frac{i\tilde{\alpha}}{k\pi} (\cos k\pi - \cos k\pi y) \\ \sin \phi \sin k\pi y \end{array} \right\}$$

In order to maximize the growth function, it is sufficient to maximize the energy

$$E(t) = \int_{-1}^1 \vec{u}(y, t) \cdot \vec{u}^*(y, t) dy$$

subject to the constraint

$$E(0) = 1.$$

Therefore, we use Lagrange multipliers to maximize the function

$$\bar{G}(t) = \int_{-1}^1 \vec{u}(y, t) \cdot \vec{u}^*(y, t) dy - \lambda \left(\int_{-1}^1 \vec{u}(y, 0) \cdot \vec{u}^*(y, 0) dy - 1 \right)$$

which requires

$$\frac{\partial \bar{G}}{\partial a_k} = 0, \quad \frac{\partial \bar{G}}{\partial b_k} = 0, \quad k = 1, 2, \dots, N.$$

The set of equations thus derived produce a $2N \times 2N$ generalized eigenvalue problem. A search over the eigenvectors gives the initial condition with initial unit energy that maximizes the function \bar{G} at time t .

To illustrate the optimization procedure, we perform the calculations for the two cases reported by Butler and Farrell (1992). The first is the computation of the two-dimensional optimal for $\tilde{\alpha} = 1.48$, $\phi = 0^\circ$ and $Re = 5,000$. In figure 8(A) we show the growth factor at $t = 14.1$ for each individual mode as well as for the optimal solution for various values of N . The convergence as $N \rightarrow \infty$ is nicely illustrated; compare to Reddy and Henningson (1993), e.g. In figure 8(B) the magnitudes of the coefficients which produce the optimum with $N = 20$ are shown. There are no surprises. Each coefficient is of reasonable size, with the largest coefficient being a factor of ten greater than the first coefficient, and not a factor of a thousand as is the case when using eigenfunction expansions. The magnitudes peak for $n = k, 7$ and 8 which could be easily predicted from the previous graphs for the responses to each individual mode. For completeness the initial velocity profiles are shown in figure 9. These are consistent with the the initial perturbation streamfunction contour plots shown in Butler and Farrell (1992). A similar calculation could be made for Couette flow but is unnecessary since the relevant information is easily determined by examination of figure 4(B). The magnitudes

of the coefficients which produce the optimal growth at $t = 8.7$ peaks between modes 3 and 4, and converge quickly as $N \rightarrow \infty$. The resulting initial condition is consistent with the initial perturbation streamfunction contour plots shown in Butler and Farrell (1992) for the same case. The second calculation is for the optimal three-dimensional disturbance. The parameters chosen are $Re = 5,000$, $\tilde{\alpha} = 2.044$ and $\phi = 90^\circ$. The initial conditions that produce a maximum growth at $t = 379$ are found. The results are shown in figure 10, and the composition of the initial conditions in terms of the modes chosen here could be easily determined from the individual responses of each mode. The key to this observation is that the chosen modes are not nearly linearly dependent as are the eigenfunctions and indeed provide a rational and easily understood basis for the calculation of arbitrary initial conditions.

It must be re-iterated that, although it is possible and conceptually easy to reproduce the optimal initial conditions that have been previously found, the maximum transient growth is only a measure of what is possible and not what will actually occur as has been the difficulty in experiments. It is at least as important to investigate whether such large growth is possible for arbitrary initial conditions. In this regard, the results presented here produce a mostly negative answer to this question. For two-dimensional disturbances in Poiseuille flow, the transient growth observed for arbitrarily chosen initial conditions using this approach are, at best, only 25% of the optimal. When considering a fixed wavelength $\tilde{\alpha}$ and a fixed obliqueness ϕ , it is seen that very large relative energy growth of the perturbation can be observed in Poiseuille flow for oblique disturbances with arbitrary velocity profiles restricted to having zero initial normal vorticity, but the relative energy growth quickly decreases when arbitrary disturbances are combined with initial normal vorticity. Similar results are found for Couette flow.

5 CONCLUSIONS

Plane Poiseuille flow and plane Couette flow in an incompressible fluid have been investigated subject to the influence of small perturbations. Instead of using the techniques of classical stability analysis or the more recent techniques involving eigenfunction expansions, the approach has been to first Fourier transform the governing disturbance equations in the streamwise and spanwise directions only and then solve the resulting partial differential equations numerically by the method of lines. Unlike traditional methods where traveling wave normal modes are assumed for solution, this approach offers a means whereby completely arbitrary initial input can be specified without having to resort to the complexity of eigenfunction expansions. Thus, arbitrary initial conditions can be imposed and the full temporal behavior, including both early time transients and the long time

asymptotics, can be determined. All of the stability data that are known for such flows can be reproduced. Finally, an optimization scheme is presented using the orthogonal Fourier series and all previous results using variational techniques and eigenfunction expansions are reproduced.

The benefit of this novel approach is clear: it can be applied to other classes of problems where only a finite number of normal modes exist, such as the boundary layer. In addition, this numerical approach has recently been successfully applied to free shear flows in an inviscid fluid (Criminale et al., 1994). These concepts are being extended to the Blasius boundary layer in an incompressible or compressible medium.

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7 FIGURE CAPTIONS

Figure 1. (A) Plot of growth rates as a function of wavenumber α . The circles corresponds to the numerically computed values from the partial differential equation, and the solid curve corresponds to the growth rate computed using the Orr-Sommerfeld equation. (B) The real part and (C) the imaginary part of the eigenfunction as a function of y for $\alpha = 1$. Results for plane Poiseuille flow with $\phi = 0^\circ$ and $Re = 10^4$.

Figure 2. (A) Plot of growth rates as a function of wavenumber α . The circles corresponds to the numerically computed values from the partial differential equation, and the solid curve corresponds to the growth rate computed using the Orr-Sommerfeld equation. (B) The real part and (C) the imaginary part of the eigenfunction as a function of y for $\alpha = 0.44$. Results for plane Poiseuille flow with $\phi = 0^\circ$ and $Re = 10^6$.

Figure 3. Plot of the growth function G as a function of time for various initial conditions. Results for plane Poiseuille flow with $\tilde{\alpha} = 1.48$, $\phi = 0^\circ$, and $Re = 5,000$.

Figure 4. Plot of the growth function G as a function of time for various initial conditions. Results for plane Couette flow with $\tilde{\alpha} = 1.21$, $\phi = 0^\circ$, and $Re = 1,000$.

Figure 5. Plot of the growth function G as a function of time for various initial conditions. Results for plane Poiseuille flow with $\tilde{\alpha} = 2.044$, $\phi = 90^\circ$, and $Re = 5,000$.

Figure 6. Plot of the growth function G as a function of time for various initial conditions. Results for plane Couette flow with $\tilde{\alpha} = 1.66$, $\phi = 90^\circ$, and $Re = 1,000$.

Figure 7. Plot of the vorticity components as a function of time for the initial conditions (A) (II,i) and (B) (II,ii). Results for plane Poiseuille flow with $\tilde{\alpha} = 2.044$, $\phi = 90^\circ$, and $Re = 5,000$.

Figure 8. (A) Plot of the growth function G at $t = 14.1$; individual mode results denoted by $*$; cumulative results from optimization procedure denoted by \circ . (B) Plot of the magnitude of the coefficients $a_k + ib_k$ from optimization procedure for $N = 20$. For plane Poiseuille flow with $\tilde{\alpha} = 1.48$, $\phi = 0^\circ$, and $Re = 5,000$.

Figure 9. Plot of velocities for plane Poiseuille flow from optimization procedure with $N = 20$, $\tilde{\alpha} = 1.48$, $\phi = 0^\circ$, and $Re = 5,000$. (A) \check{u}_r (solid), \check{u}_i (dashed) at $t = 0$; (B) \check{v}_r (solid), \check{v}_i (dashed) at $t = 0$; (C) \check{u}_r (solid), \check{u}_i (dashed) at $t = 14.1$; (D) \check{v}_r (solid), \check{v}_i (dashed) at $t = 14.1$.

Figure 10. (A) Plot of the growth function G at $t = 379.0$; individual mode results denoted by $*$; cumulative results from optimization procedure denoted by \circ . (B) Plot of the magnitude of the coefficients $a_k + ib_k$ from optimization procedure for $N = 20$. For plane Poiseuille flow with $\tilde{\alpha} = 2.044$, $\phi = 90^\circ$, and $Re = 5,000$.