Maximum Principles and Application to the Analysis of an Explicit Time Marching Algorithm

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Abstract

In this paper we develop local and global estimates for the solution of advection-diffusion problems. We then study the convergence properties of a Time Marching Algorithm solving advection-diffusion problems on two domains using incompatible discretizations. This study is based on a De-Giorgi-Nash maximum principle.

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1 Introduction

Domain decomposition methods have recently become an efficient strategy for solving large scale problems on parallel computers ([1], [2], [3], [4], [5], [6]). Nevertheless, they can also be used in order to couple different models [11], [18], [19] and [21]. In this paper we will examine a domain decomposition strategy which can be applied to such situations.

This approach was introduced in order to solve several difficulties that occur in fluid mechanics. In particular, our aim is to introduce several subdomains in order to do one of the following :

- Solve different problems on each subdomain.
- Use different kinds of approximation methods on each subdomain [7].
- Use "local refinement techniques" or "mesh adaptive techniques", locally, per subdomain ([10]).

The subdomains fully overlap and the coupling is achieved through "friction" forces acting on the internal boundary of the domain, these friction forces being updated by an explicit time marching algorithm.

Several versions of this methodology have been studied in [15]. In [15] the emphasis was on the implicit time discretization version of this algorithm, we focus in this paper on the explicit version of this methodology. The theoretical study of our method will be done on an advection-diffusion problem, which will serve as our model problem. The analysis will be made at the continuous level, independently of any (space) discretization strategy, which means that the derived results will be mesh independent.

In the next section we develop a maximum principle for general second-order elliptic problem based on the De-Giorgi-Nash theory. In sections 2 and 3, we develop estimates for the solution of the advection-diffusion problems, respectively, with Dirichlet-Neumann and Dirichlet boundary conditions. These results are based on the maximum principle of section 2. We then apply these tools to the analysis of an explicit time marching algorithm. We also study a fixed point method for the implicit time marching algorithm of [15]. Practical applications of the time marching algorithm to real life CFD problems can be found in [14], [19], [20], and [21].

2 Local estimates

In this section we shall establish a maximum principle for an arbitrary elliptic operator of second order. These tools are central to the development of our theory in order to derive the convergence analysis of the explicit time marching algorithm described in section 5.1.

Let L be an operator of the form

$$Lu = a^{ij}(x)D_{ij}u + b^{i}(x)D_{i}u + c(x)u,$$

for any u in $W^{2,n}(\Omega)$, with Ω a bounded domain of $\mathbb{R}^n$. The coefficients a^{ij}, b^i and $c, i, j = 1, ..., n$ are defined on Ω. As usual, the repeated indices indicate a summation from 1 to n.

We suppose that the operator L is strictly elliptic in Ω in the sense that the matrix $\mathcal{A}$ of coefficients $[a^{ij}]$ is strictly positive everywhere in Ω. Let λ and Λ denote respectively the smallest and the largest eigenvalue of $\mathcal{A}$. Let $\mathcal{D}$ denote the determinant of the matrix $\mathcal{A}$ and $\mathcal{D}^* = \mathcal{D}^{1/n}$. We have

$$0 < \lambda \le \mathcal{D}^* \le \Lambda.$$

We suppose in addition that the coefficients a^{ij}, b^i and c are bounded in Ω, and that there exist two positive real numbers γ and δ such that :

$$\Lambda/\lambda \le \gamma, \qquad (L \text{ is uniformly elliptic}) \tag{1}$$

$$(|b|/\lambda)^2 \le \delta. \tag{2}$$

Now, we are in a position to state the principal result of this section, proved in annex.

Theorem 2.1 *Let $u \in W^{2,n}(\Omega)$ and suppose that $Lu \ge f$ with $f \in L^n(\Omega)$ and $c \le 0$. Then for all spheres $B = B_{2R}(y)$ of center y and radius $2R$ included in Ω and for all $p > 0$, we have :*

$$sup_{B_R(y)} u \le C_R\{(\frac{1}{|B|}\int_B (u^+)^p)^{1/p} + \frac{R}{\lambda}\|f\|_{n,B}\}, \tag{3}$$

where the constant C_R depends on $(n, \gamma, \delta R^2, p)$, but is independent of c. Above $u^+ = max(u, 0)$.

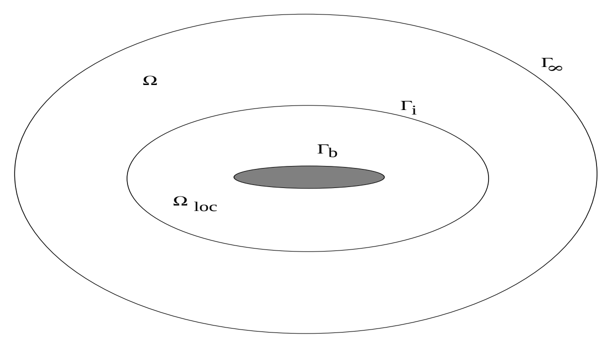

Figure 1: Description of the computational domain.

Remark 2.1 *The statement of the same theorem can be found in [12], under the assumption*

$$|c|/\lambda \leq \delta. \tag{4}$$

However, there the constant C_R depends indirectly on c through δ. That is exactly what we want to avoid, since we would like this constant to be independent of c (see section 5.1).

3 First fundamental estimate

Let Ω_{loc} be a connected domain of $\mathbb{R}^n$ with $\Omega_{loc} \subset \Omega$ (Figure 1). The boundaries of the two subdomains are defined as follows:

$$\Gamma_b = \partial\Omega \cap \partial\Omega_{loc} \text{ (internal boundary)},$$

$$\Gamma_i = \partial\Omega_{loc} \cap \Omega \text{ (interface)},$$

$$\Gamma_\infty = \partial\Omega \backslash \Gamma_b \text{ (farfield boundary)}.$$

We denote by n the external unit normal vector to $\partial\Omega$ or $\partial\Omega_{loc}$.
Let V be a given velocity field of an inviscid incompressible flow such that:

$$\begin{cases} \text{div}V = 0 \text{ in } \Omega, \\ V.n = 0 \text{ on } \Gamma_b. \end{cases} \tag{5}$$

We shall derive an estimate for the solution of the following Dirichlet-Neumann problem:

$$\begin{aligned} \mathcal{L}v &= -\nu\Delta v + V\cdot\nabla v + \frac{1}{\tau}v \;\text{ in }\; \Omega, && (6)\\ v &= 0 \;\text{ on }\; \Gamma_\infty, && (7)\\ \frac{\partial v}{\partial n} &= g \;\text{ on }\; \Gamma_b, && (8)\end{aligned}$$

where the function g is given in $H^{-1/2}(\Gamma_b)$ and the coefficient τ is strictly positive, and ν is the diffusion coefficient. Let W be the sub-space of $H^1(\Omega)$ defined by

$$W = \{w \in H^1(\Omega) |\; w = 0 \text{ on } \Gamma_\infty\} \tag{9}$$

We then define the following bilinear forms on W

$$a(v,w) = \int_\Omega \nu\nabla v\nabla w + \int_\Omega \operatorname{div}(Vv)w, \tag{10}$$

$$(v,w) = \int_\Omega vw. \tag{11}$$

The first basic problem associated to (6)-(8), can be written as follows: Find $v \in W$ satisfying:

$$a(v,w) + (1/\tau)(v,w) = \int_{\Gamma_b} gw\,d\Gamma, \;\; \forall w \in W. \tag{12}$$

Moreover, we assume that the coefficients ν and τ satisfy the following relation:

$$\nu\tau \leq 1. \tag{13}$$

This hypothesis is not necessary but simplifies the proofs to come. Moreover, it is not restrictive, since we would like the convergence for small τ (see section 5.1).

Let d denote the overlapping distance as described in Figure 2. Let β be a real number such that

$$0 < \beta < 3\sqrt{\nu}/d,$$

and set

$$k = \beta/(\nu\sqrt{\tau}).$$

The first basic result states the global H^1 estimate of the solution of the first basic problem (12) in terms of the boundary data g.

Lemma 3.1 *There exists a constant c_o such that:*

$$\|v\|_{1,\Omega} \leq (c_o/\nu)\|g\|_{-1/2,\Gamma_b}. \tag{14}$$

Proof of lemma 3.1

By using the relation (5) we have the following equality:

$$\begin{aligned}
\int_\Omega v\mathrm{div}(Vv) &= 1/2\int_\Omega \mathrm{div}(Vv^2) \\
&= 1/2\int_\Gamma V.nv^2 \\
&= 0,\ \forall\ v\in W.
\end{aligned}$$

Choosing $w = v$ in (12), we then obtain

$$\int_\Omega \{\nu|\nabla v|^2 + (1/\tau)v^2\} = \int_{\Gamma_b} gv. \tag{15}$$

From this equality we deduce the following estimate:

$$\nu\|v\|_{1,\Omega}^2 \leq \|g\|_{-1/2,\Gamma_b}\|v\|_{1/2,\Gamma_b}.$$

The application of the trace theorem yields the estimate (14), which implies in particular

$$\|v\|_{0,\Omega} \leq (c_o/\nu)\|g\|_{-1/2,\Gamma_b}. \tag{16}$$

■

Let Ω_i be the subdomain of width $\frac{d}{3}$ with external boundary Γ_i as described in Figure 2. Let $K_y = B_{\frac{d}{3}}(y)$ be the sphere of center y and radius $\frac{d}{3}$. There exist $y_1,\ldots,y_l$ belonging to Ω_i such that

$$\Omega_{2i} = \cup_{y\in\Omega_i} B_{\frac{d}{6}}(y) \subset \cup_{j=1}^{l} K_{y_j}.$$

We define then K by setting

$$K = \cup_{j=1}^{l} K_{y_j}.$$

The next lemma states the local estimate of the solution v of the first basic problem (12).

Lemma 3.2 *There exists a constant c_1 such that:*

$$\|v\|_{\infty,K} \leq c_1 \|v\|_{0,\Omega}. \tag{17}$$

where c_1 is a constant depending only on $\nu, \gamma, \delta d^2$ and $(3/2d)^{n/2}$.

Proof of lemma 3.2

The operator

$$L = -\mathcal{L}$$

satisfies the assumptions of the theorem 2.1, with $c = -1/\tau$ and $f = 0$. Applying then this theorem with $p = 2,\ y \in \Omega_i$ we obtain

$$\|v\|_{\infty,Ky} \leq c_1 \|v\|_{0,B_{2d/3}(y)}.$$

Therefore

$$\|v\|_{\infty,Ky} \leq c_1 \|v\|_{0,\Omega}, \tag{18}$$

where c_1 is a constant depending only on $\nu, \gamma, \delta d^2$ and $(3/2d)^{n/2}$.

Applying the relation (18) to each K_{y_j} we obtain:

$$\|v\|_{\infty,K} \leq \sup_{j=1,\ldots,l} c_{1j} \|v\|_{0,\Omega}.$$

Setting $c_1 = \sup_{j=1,\ldots,l} c_{1j}$, we finally have

$$\|v\|_{\infty,K} \leq c_1 \|v\|_{0,\Omega}. \tag{19}$$

And the lemma is proved.

■

We shall now establish other local estimates for the solution v of the first basic problem. For any M_i in Ω_i, we introduce (see Figure 2):

- B_i = the ball centered on M_i of radius $d/6$,
- $v_i = exp[k(r^2 - d^2/36)]\|v\|_{\infty,\partial B_i}$.

We then have:

Lemma 3.3 *The solution v of the first basic problem satisfies:*

$$|v(M_i)| \leq \exp(-kd^2/36)\|v\|_{\infty,\partial B_i},\ \forall M_i \in \Omega_i. \tag{20}$$

Proof of lemma 3.3

The operator $\mathcal{L}$ applied to v_i, can be written in polar coordinates (with $r = M_iM$) :

$$Lv_i = 4(-k^2\nu r^2 - k\nu + \frac{k}{2}V.e_r r + \frac{1}{4\tau})v_i.$$

Therefore

$$Lv_i \geq 4(-k^2\nu r^2 - \frac{k}{2}|V.e_r|r + (\frac{1}{4\tau} - k\nu))v_i. \tag{21}$$

We set then

$$\varphi(r,k) = a(k)r^2 + b(k)r + c(k), \tag{22}$$

with

$$\begin{aligned} a(k) &= -k^2\nu \\ b(k) &= -\frac{k}{2}|V.e_r| \\ c(k) &= \frac{1}{4\tau} - k\nu. \end{aligned}$$

We seek to satisfy the following relation :

$$0 \leq \inf \varphi(r,k) \text{ for } 0 \leq r \leq \frac{d}{6}.$$

As $\varphi(r,k)$ decreases on $\mathbb{R}^+$, this will be satisfied iff

$$\varphi(d/6) \geq 0,$$

i.e. iff

$$-\frac{k^2\nu d^2}{36} - \frac{kd\|V\|}{12} + \frac{1}{4\tau} - k\nu \geq 0.$$

We replace k by its value. Therefore, we have to satisfy

$$-\frac{\beta^2 d^2}{(36\nu\tau)} - \frac{\beta d\|V\|}{12\nu\sqrt{\tau}} + \frac{1}{4\tau} - \frac{\beta\nu}{\nu\sqrt{\tau}} \geq 0.$$

Multiplying by $\sqrt{\tau}$, it follows that

$$\frac{1}{4\sqrt{\tau}}(1-\frac{\beta^2 d^2}{9\nu}) \geq \beta(1+\frac{d\|V\|}{12\nu}).$$

The constraint $\beta < 3\sqrt{\nu}/d$, finally yields after division

$$\varphi(r,k) \geq 0 \ \text{ iff } \ \frac{1}{4\sqrt{\tau}} \geq \beta[1+\frac{d\|V\|}{12\nu}][1-\frac{\beta^2 d^2}{9\nu}]^{-1}. \tag{23}$$

From the relation (21) and the previous calculation, we deduce that for $\beta < 3\sqrt{\nu}/d$ and τ satisfying the above inequality, we have

$$Lv_i \geq 0 = Lv.$$

In addition, by construction

$$v_i \geq v \text{ on } \partial B_i.$$

Consequently, by using the maximum principle we obtain the following relation :

$$v \leq v_i \ \text{ in } \ B_i.$$

In particular

$$v(M_i) \leq \exp(-kd^2/36)\|v\|_{\infty,\partial B_i}.$$

We do the same for $-v$, and finally we have

$$|v(M_i)| \leq \exp(-kd^2/36)\|v\|_{\infty,\partial B_i},\ \forall M_i \in \Omega_i. \tag{24}$$

■

Let Ω_∞ be defined by $\Omega_\infty = \Omega \setminus \Omega_{loc}$. The next result establishes an H^1 estimate of the solution v of the first basic problem on the domain Ω_∞.

Lemma 3.4 *There exists a constant c_2 such that:*

$$\|v\|_{1,\Omega_\infty} \leq \|v\|_{\infty,\Omega_i}\sqrt{c_2/d}\left(1+\frac{\|V\|}{\nu}\sqrt{d/c_2}\right)^{1/2}. \tag{25}$$

Proof of lemma 3.4

Let $\xi \in H^1(\Omega)$ be such that :

$$\begin{cases} \xi = 1 \ \text{ in } \ \Omega_\infty, \\ \operatorname{supp}\xi \subset \Omega_i \cup \Omega_\infty. \end{cases}$$

We have using (12):

$$\int_\Omega (-\nu\Delta v + \operatorname{div}(Vv) + v/\tau)\xi^2 v = 0. \tag{26}$$

By using the Green's formula we deduce :

$$\int_\Omega -\nu\Delta v\xi^2 v = \int_\Omega \nu|\nabla(\xi v)|^2 - \int_\Omega \nu|\nabla\xi|^2 v^2. \tag{27}$$

On the other hand, we have :

$$\int_\Omega \operatorname{div}(Vv)\xi^2 v = \int_\Omega \operatorname{div}(V\xi^2 v^2/2) - \int_\Omega V.\nabla\xi\xi v^2. \tag{28}$$

Using the relations (27) and (28), (26) becomes

$$\begin{aligned}
0 &= \int_\Omega (\nu|\nabla(\xi v)|^2 + \operatorname{div}(V\xi^2 v^2/2) + \xi^2 v^2/\tau) - \int_\Omega (\nu v^2|\nabla\xi|^2 + v^2\xi V\cdot\nabla\xi) \\
&= \int_\Omega \nu(|\nabla(\xi v)|^2 + |\xi v|^2) + \int_\Omega (1/\tau - \nu)\xi^2 v^2 - \int_\Omega (\nu v^2|\nabla\xi|^2 + v^2\xi V.\nabla\xi) \\
&= \int_{\Omega_\infty} \nu(|\nabla v|^2 + |v|^2) + \int_{\Omega_i} \nu(|\nabla(\xi v)|^2 + |\xi v|^2) + \int_\Omega (1/\tau - \nu)\xi^2 v^2 \\
&\quad - \int_{\Omega_i} (\nu v^2|\nabla\xi|^2 + v^2\xi V.\nabla\xi).
\end{aligned}$$

Hence, we obtain :

$$\nu\|v\|^2_{1,\Omega_\infty} + \int_{\Omega_i} \nu(|\nabla(\xi v)|^2 + |\xi v|^2) + \int_\Omega (1/\tau - \nu)\xi^2 v^2 =$$

$$\int_{\Omega_i} (\nu v^2|\nabla\xi|^2 + v^2\xi V\cdot\nabla\xi).$$

The relation (13) then yields

$$\begin{aligned}
\nu\|\xi v\|^2_{1,\Omega_\infty\cup\Omega_i} &\leq \int_{\Omega_i}(\nu v^2|\nabla\xi|^2 + v^2\xi V.\nabla\xi) & (29)\\
&\leq \|v\|^2_{\infty,\Omega_i}\int_{\Omega_i}(\nu|\nabla\xi|^2+\xi V.\nabla\xi) & (30)\\
&\leq \|v\|^2_{\infty,\Omega_i}(\nu|\xi|^2_{1,\Omega_i}+\|\xi\|_{0,\Omega_i}\|V\||\xi|_{1,\Omega_i}) & \\
&\leq \|v\|^2_{\infty,\Omega_i}|\xi|^2_{1,\Omega_i}(\nu+\|\xi\|_{0,\Omega_i}\|V\|/|\xi|_{1,\Omega_i}). & (31)
\end{aligned}$$

If we take ξ such that

$$\|\xi\|_{0,\Omega_i}\leq 1,$$

$$|\xi|^2_{1,\Omega_i} = c_2/d,$$

where c_2 is a constant, (31) then becomes

$$\|v\|_{1,\Omega_\infty}\leq\|v\|_{\infty,\Omega_i}\sqrt{c_2/d}\left(1+\frac{\|V\|}{\nu}\sqrt{d/c_2}\right)^{1/2}, \tag{32}$$

which concludes the proof.

■

Now we are in a position to state the main result of this section.

Theorem 3.1 *Let v be the solution of the first basic problem (12). If τ is sufficiently small, we have*

$$\begin{aligned}
\|v\|_{1/2,\Gamma_i} &\leq C_1\sqrt{C_2/d}\left(1+\frac{1}{\nu}\|V\|_\infty\sqrt{d/C_2}\right)^{1/2}\\
&\quad (1/\nu)exp(-kd^2/36)\|g\|_{-1/2,\Gamma_b},
\end{aligned}$$

where C_1 and C_2 are constants, with C_1 depending only on d,ν,γ and δ, but not on τ.

Proof of theorem 3.1

The proof of this theorem is based on the above lemmas. Since $\partial B_i \subset K$, We have

$$\|v\|_{\infty,\partial B_i} \leq \|v\|_{\infty,K}, \tag{33}$$

The lemma 3.2 then implies

$$\|v\|_{\infty,\partial B_i} \leq c_1\|v\|_{0,\Omega}. \tag{34}$$

Using the lemma 3.3 and the above estimate we obtain:

$$|v(M_i)| \leq \exp(-kd^2/36)c_1\|v\|_{0,\Omega},\ \forall M_i \in \Omega_i.$$

Consequently we have

$$\|v\|_{\infty,\Omega_i} \leq \exp(-kd^2/36)c_1\|v\|_{0,\Omega}. \tag{35}$$

Applying the lemma 3.1 we obtain:

$$\|v\|_{\infty,\Omega_i} \leq \frac{c_1 c_o}{\nu}\exp(-kd^2/36)\|g\|_{-1/2,\Gamma_b}. \tag{36}$$

The application of the lemma 3.4 then yields:

$$\|v\|_{1,\Omega_\infty} \leq c_o c_1\sqrt{c_2/d}\left(1+\frac{\|V\|}{\nu}\sqrt{d/c_2}\right)^{1/2} \tag{37}$$
$$(1/\nu)\exp(-kd^2/36)\|g\|_{-1/2,\Gamma_b}.$$

To conclude we use the trace theorem which yields

$$\|v\|_{1/2,\Gamma_i} \leq c_3\|v\|_{1,\Omega_\infty}.$$

Consequently, we have the final estimate:

$$\begin{aligned}\|v\|_{1/2,\Gamma_i} \quad \leq \quad & c_o c_1 c_3\sqrt{c_2/d}(1+\frac{\|V\|}{\nu}\sqrt{d/c_2})^{1/2} \\ & (1/\nu)\exp(-kd^2/36)\|g\|_{-1/2,\Gamma_b},\end{aligned}$$

which corresponds to our theorem with $C_1 = c_o c_1 c_3$ and $C_2 = c_2$.

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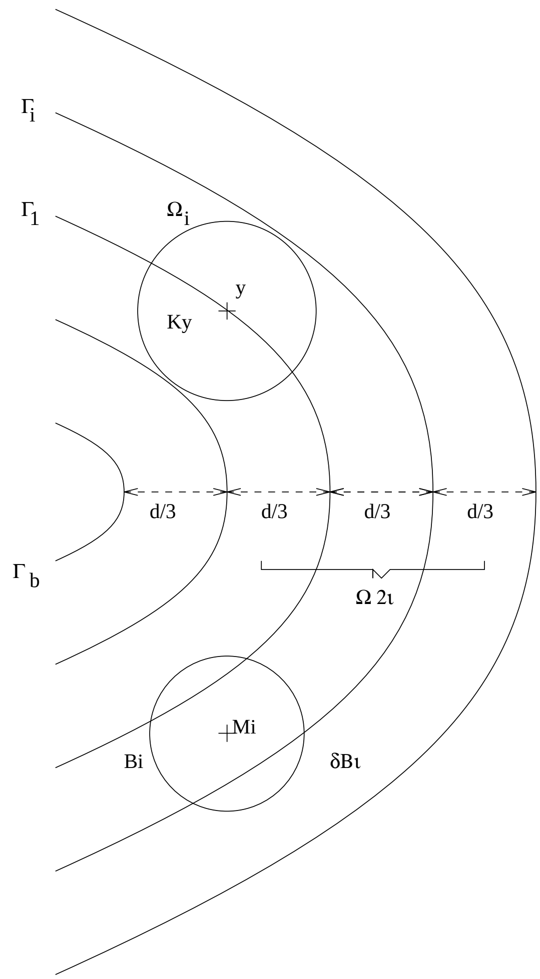

Figure 2: Description of the Domain Ω_{loc} and of the splitting used in the majoration of the local solution.

4 Second fundamental estimate

In this section we shall derive an estimate of the solution of the following Dirichlet problem:

$$\begin{aligned}
\mathcal{L}v &= -\nu\Delta v + V\cdot\nabla v + \frac{1}{\tau}v \text{ in } \Omega_{loc}, &(38)\\
v &= h, \text{ on } \Gamma_i, &(39)\\
v &= 0, \text{ on } \Gamma_b, &(40)
\end{aligned}$$

where the function h is given in $H^{1/2}(\Gamma_i)$, the coefficient τ is strictly positive, and ν is the diffusion coefficient. The velocity field V is given by the relation (5). Let W be the subspace of $H^1(\Omega_{loc})$ defined by

$$W = \{w \in H^1(\Omega_{loc}) |\ w = 0 \text{ on } \Gamma_b\}.$$

We then define the following bilinear forms in W:

$$a(v,w) = \nu\int_{\Omega_{loc}} \nabla v.\nabla w + \int_{\Omega_{loc}} \text{div}(Vv)w, \tag{41}$$

$$(v,w) = \int_{\Omega_{loc}} vw, \tag{42}$$

with v and w in W. The second basic problem associated to (38)-(40) corresponds to the following problem:
Find $v \in W$ such that

$$a(v,w) + (1/\tau)(v,w) = \int_{\Gamma_i} \nu\frac{\partial v}{\partial n}w, \quad \forall w \in W, \tag{43}$$

$$v|_{\Gamma_i} = h, \tag{44}$$

where h is given in $H^{1/2}(\Gamma_i)$. We first have the following lemma:

Lemma 4.1 *For τ sufficiently small, we have*

$$a(w,w) + (1/\tau)(w,w) \geq (\nu/2)\|w\|^2_{1,\Omega_{loc}}, \quad \forall w \in W.$$

Proof of lemma 4.1:

Under the hypothesis $1/\tau \geq \nu/2 + (1/2\nu)\|V\|_\infty^2$, and using the Cauchy-Schwarz inequality, we obtain :

$$\begin{aligned}
a(v,v) + (1/\tau)(v,v) &= \int_{\Omega_{loc}} \nu \nabla v.\nabla v + \int_{\Omega_{loc}} V.\nabla v v + (1/\tau)\int_{\Omega_{loc}} v^2 \\
&\geq \nu\|\nabla v\|_{0,2}^2 + (1/\tau)\|v\|_{0,2}^2 - \|V\|_\infty \|\nabla v\|_{0,2}\|v\|_{0,2} \\
&\geq \nu\|\nabla v\|_{0,2}^2 + (1/\tau)\|v\|_{0,2}^2 - (\nu/2)\|\nabla v\|_{0,2}^2 \\
&\quad -(1/2\nu)\|V\|_\infty^2\|v\|_{0,2}^2 \\
&\geq (\nu/2)(\|\nabla v\|_{0,2}^2 + \|v\|_{0,2}^2).
\end{aligned}$$

■

We will also make the simplifying assumption (13). We first establish a global estimate for the solution of the second basic problem.

Lemma 4.2 *The solution v of the second basic problem (43)-(44) satisfies:*

$$\|v\|_{1,\Omega_{loc}} \leq 2(1+\tau^{-2})^{1/2}\left(1 + \frac{1+\|V\|_\infty^2}{\nu^2}\right)^{1/2} \|h\|_{1/2,\Gamma_i} \tag{45}$$

Proof of lemma 4.2:

Choosing $w = v$ in (43) we obtain :

$$\nu \int_{\Omega_{loc}} |\nabla v|^2 + \int_{\Omega_{loc}} (\mathrm{div}(Vv)v + (1/\tau)v^2) = \int_{\Gamma_i} \nu \frac{\partial v}{\partial n} h. \tag{46}$$

The lemma 4.1 then yields

$$\|v\|_{1,\Omega_{loc}}^2 \leq 2\|\partial v/\partial n\|_{-1/2,\Gamma_i}\|h\|_{1/2,\Gamma_i}. \tag{47}$$

We shall now establish an estimate of $\|\partial v/\partial n\|_{-1/2,\Gamma_i}$. Combining (43) and (5) we obtain:

$$\int_{\Gamma_i} \frac{\partial v}{\partial n} w = \int_{\Omega_{loc}} (\nabla v \nabla w + (1/\nu)V.\nabla v w + \frac{1}{\nu\tau} v w).$$

Therefore, for any w in W, we have

$$
\begin{aligned}
|\int_{\Gamma_i} \frac{\partial v}{\partial n} w| &\leq \|\nabla v\|_{0,\Omega_{loc}} \|\nabla w\|_{0,\Omega_{loc}} + (1/\nu)\|V\|_\infty \|\nabla v\|_{0,\Omega_{loc}} \|w\|_{0,\Omega_{loc}} \\
&\quad + \frac{1}{\nu\tau} \|v\|_{0,\Omega_{loc}} \|w\|_{0,\Omega_{loc}} \\
&\leq (\|\nabla v\|^2_{0,\Omega_{loc}} + (1/\nu^2)\|V\|^2_\infty \|\nabla v\|^2_{0,\Omega_{loc}} + (1/\nu^2)\|v\|^2_{0,\Omega_{loc}})^{1/2} \\
&\quad (\|\nabla w\|^2_{0,\Omega_{loc}} + \|w\|^2_{0,\Omega_{loc}} + (1/\tau^2)\|w\|^2_{0,\Omega_{loc}})^{1/2} \\
&\leq [1 + \frac{(1+\|V\|^2_\infty)}{\nu^2}]^{1/2} \|v\|_{1,\Omega_{loc}} (1+\tau^{-2})^{1/2} \|w\|_{1,\Omega_{loc}}.
\end{aligned}
$$

The trace theorem then yields

$$
\|\partial v/\partial n\|_{-1/2,\Gamma_i} \leq (1+\tau^{-2})^{1/2} \left(1 + \frac{1+\|V\|^2_\infty}{\nu^2}\right)^{1/2} \|v\|_{1,\Omega_{loc}}. \tag{48}
$$

Combining now the relations (47) and (48) we have

$$
\|v\|_{1,\Omega_{loc}} \leq 2(1+\tau^{-2})^{1/2} \left(1 + \frac{1+\|V\|^2_\infty}{\nu^2}\right)^{1/2} \|h\|_{1/2,\Gamma_i} \tag{49}
$$

and hence in particular

$$
\|v\|_{0,\Omega_{loc}} \leq 2(1+\tau^{-2})^{1/2} \left(1 + \frac{1+\|V\|^2_\infty}{\nu^2}\right)^{1/2} \|h\|_{1/2,\Gamma_i}. \tag{50}
$$

■

Let $K_y = B_{d/4}(y)$ be the sphere centered on y and of radius $d/4$, with y belonging to Γ_V (see Figure 3). By construction, Γ_V is the center surface of Ω_{loc} and Ω_i is the subdomain of width $\frac{d}{6}$ centered on Γ_V .
We have the following lemma:

Lemma 4.3 *There exists a constant c_1 such that:*

$$
\|v\|_{\infty,K} \leq c_1 \|v\|_{0,\Omega_{loc}}. \tag{51}
$$

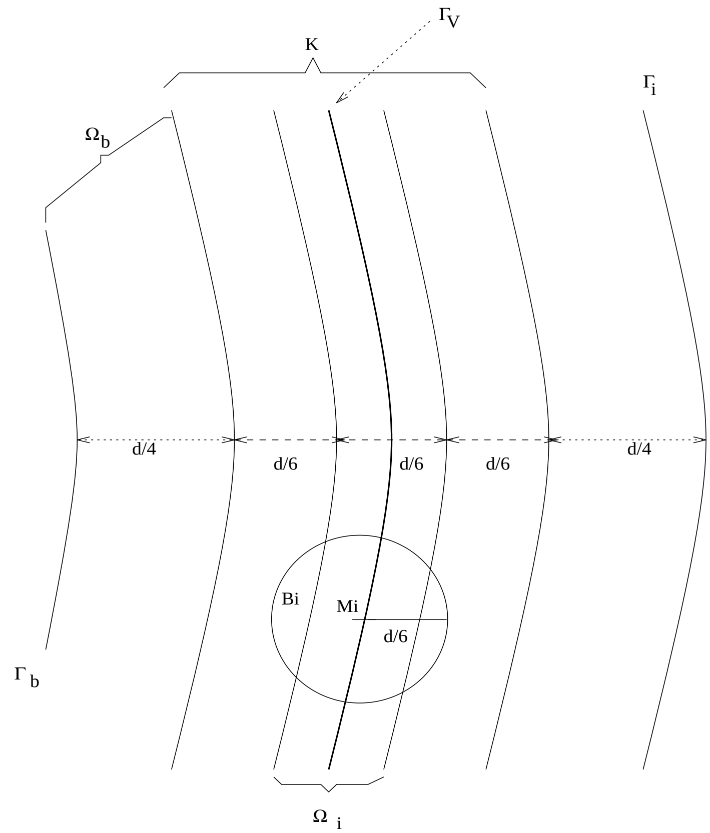

Figure 3: Description of the local domain Ω_{loc} and of the splitting used in the majoration of the global solution.

Proof of lemma 4.3:

Following the same argument as in the proof of the lemma 3.2 we obtain:

$$\|v\|_{\infty,K_y} \leq c_1 \|v\|_{0,\Omega_{loc}}, \tag{52}$$

where c_1 is a constant depending only on d, ν, γ and δ. On the other hand there exist $y_1, \ldots, y_l$ in Ω_i such that

$$\Omega_{2i} = \bigcup_{y \in \Omega_i} B_{\frac{d}{6}}(y) \subset \bigcup_{j=1}^{l} K_{y_j} = K.$$

By applying the relation (52) to each K_{y_j}, we obtain

$$\|v\|_{\infty,K} \leq \sup_{j=1,\ldots,l} c_{1j} \|v\|_{0,\Omega_{loc}} = c_1 \|v\|_{0,\Omega_{loc}}. \tag{53}$$

■

Next we shall establish another local estimate for the solution of the second basic problem (43)-(44). For any $M_i \in \Omega_i$, we introduce (see Figure 3):

- a ball B_i centered on M_i and of radius $d/6$,

- the function $v_i = \exp[k(r^2 - d^2/36)]\|v\|_{\infty,\partial B_i}$.

We then have:

Lemma 4.4 *The solution v of the second basic problem (43)-(44) satisfies:*

$$|v(M_i)| \leq exp[-kd^2/36]\|v\|_{\infty,\partial B_i}. \tag{54}$$

Proof of lemma (4.4):

By construction of k (see the previous section), $\varphi(r,k)$ is positive for all $r \in [0, d/6]$. Then by following the same argument as in the proof of the lemma 3.3 we obtain the inequality (54).

■

Let Ω_b be the subdomain of Ω_{loc} described in the Figure 3. The H^1 global estimate of the solution of the second basic problem, is obtained in the next lemma.

Lemma 4.5 *The solution v of the second basic problem (43)-(44) satisfies:*

$$\|v\|_{1,\Omega_b \cup \Omega_i} \leq \|v\|_{\infty,\Omega_i} \sqrt{c_2/d} \left(1 + \frac{\|V\|_\infty}{\nu} \sqrt{d/c_2}\right)^{1/2}. \tag{55}$$

Proof of lemma (4.5):

Consider $\xi \in H^1(\Omega_{loc})$, such that:

$$\left\{ \begin{array}{lcl} \xi & = & 1 \quad \text{in} \quad \Omega_b, \\ \text{supp}\xi \subset \Omega_b \cup \Omega_i & & \end{array} \right. \tag{56}$$

Choosing $w = \xi^2 v$ in (43) we obtain :

$$\int_{\Omega_{loc}} (-\nu \Delta v + \text{div}(Vv) + (v/\tau))\xi^2 v = 0. \tag{57}$$

Similarly to the proof of the lemma 3.4 we obtain:

$$\nu \|\xi v\|_{1,\Omega_b \cup \Omega_i} \leq \int_{\Omega_i} (\nu v^2 |\nabla \xi|^2 + v^2 \xi V.\nabla \xi).$$

Choosing ξ such that

$$\|\xi\|_{0,\Omega_i} \leq 1$$

and

$$|\xi|_{1,\Omega_i}^2 = c_2/d,$$

we finally obtain as in the proof of the lemma 3.4 the inequality (55).

■

Finally, the main result of this section is presented in the following theorem:

Theorem 4.1 *For τ sufficiently small, the solution v of the problem (43)-(44) satisfies:*

$$\begin{aligned} \|\partial v/\partial n\|_{-1/2,\Gamma_b} \quad \leq \quad & C_1\sqrt{C_2/d}\left(1+\frac{1+\|V\|_\infty^2}{\nu^2}\right) \\ & \left(1+\frac{\|V\|_\infty}{\nu}\sqrt{d/C_2}\right)^{1/2} \\ & (1+1/\tau^2)exp(-kd^2/36)\|h\|_{1/2,\Gamma_i}, \end{aligned} \tag{58}$$

where C_1 and C_2 are constants with C_1 depending only on d, v, ν and δ.

Proof of theorem 4.1:

Since $\partial B_i \subset K$ by construction, the lemmas 4.3 and 4.4 imply:

$$\|v\|_{\infty,\Omega_i} \leq \exp(-kd^2/36)c_1\|v\|_{0,\Omega_{loc}}. \tag{59}$$

Furthermore by using the lemma 4.2 it follows:

$$\begin{aligned} \|v\|_{\infty,\Omega_i} \leq 2 & \left(1+\frac{1+\|V\|_\infty^2}{\nu^2}\right)^{1/2} \\ & c_1(1+1/\tau^2)^{1/2}\exp(-kd^2/36)\|h\|_{1/2,\Gamma_i}. \end{aligned} \tag{60}$$

By using the lemma 4.5 we then obtain:

$$\begin{aligned} \|v\|_{1,\Omega_b\cup\Omega_i} \leq 2 & \left(1+\frac{1+\|V\|_\infty^2}{\nu^2}\right)^{1/2} \\ & c_1\sqrt{c_2/d}\left(1+\frac{\|V\|_\infty}{\nu}\sqrt{d/c_2}\right)^{1/2} \\ & (1+1/\tau^2)^{1/2}\exp(-kd^2/36)\|h\|_{1/2,\Gamma_i}. \end{aligned} \tag{61}$$

Before concluding we shall establish an estimate of the term

$$\|\partial v/\partial n\|_{-1/2,\Gamma_b}.$$

Choosing w such that:

$$w \in H^1(\Omega_{loc}), \text{ with } w = 0 \text{ on } \partial\Omega_b \cap \partial\Omega_i,$$

and using (43) we obtain:

$$\int_{\Omega_b} (-\nu\Delta v + \mathrm{div}(Vv) + v/\tau)w = 0.$$

Applying the Green's formula and using (5), we obtain:

$$\int_{\Gamma_b} \frac{\partial v}{\partial n} w = \int_{\Omega_b} (\nabla v \nabla w + (1/\nu)V.\nabla v w + \frac{1}{\nu\tau} v w).$$

Similarly to the proof of the lemma 4.2 we obtain the following inequality:

$$\|\partial v/\partial n\|_{-1/2,\Gamma_b} \le (1 + 1/\tau^2)^{1/2} \left(1 + \frac{1 + \|V\|_\infty^2}{\nu^2}\right)^{1/2} \|v\|_{1,\Omega_b}. \tag{62}$$

The completion of the proof of the theorem results from the combination of the relation (61) with (62).

∎

5 Convergence analysis of the explicit time marching algorithm

Consider the following elliptic problem:

$$\left\{ \begin{array}{rcl} \dfrac{\phi}{\tau} + V.\nabla\phi - \nu\Delta\phi &=& 0 \text{ in } \Omega, \\ \phi &=& \phi_\infty \text{ on } \Gamma_\infty, \\ \phi &=& 0 \text{ on } \Gamma_b, \end{array} \right. \tag{63}$$

that we would like to solve by the fundamental algorithm of [15]. This algorithm can be written in this case as

- set $\phi_{loc}^o = \phi_{ol}$ and $\phi^o = \phi_o$.

- then, for $n \geq 0, \phi_{loc}^n$ and ϕ^n being known,

solve

$$\left\{ \begin{array}{rcl} \dfrac{\phi_{loc}^{n+1}}{\tau} + V.\nabla\phi_{loc}^{n+1} - \nu\Delta\phi_{loc}^{n+1} & = & 0 \text{ in } \Omega_{loc}, \\ \phi_{loc}^{n+1} & = & \phi^n \text{ on } \Gamma_i, \\ \phi_{loc}^{n+1} & = & 0 \text{ on } \Gamma_b, \end{array} \right. \tag{64}$$

$$\left\{ \begin{array}{rcl} \dfrac{\phi^{n+1}}{\tau} + V.\nabla\phi^{n+1} - \nu\Delta\phi^{n+1} & = & 0 \text{ in } \Omega, \\ \phi^{n+1} & = & \phi_\infty \text{ on } \Gamma_\infty, \\ \nu\dfrac{\partial\phi^{n+1}}{\partial n} & = & \nu\dfrac{\partial\phi_{loc}^{n+1}}{\partial n} \text{ on } \Gamma_b. \end{array} \right. \tag{65}$$

We shall show in this section that this algorithm converges, and the converged solution corresponds to the solution of the initial problem (63). More precisely we have the following theorem.

Theorem 5.1 *For τ sufficiently small, ϕ being the solution of the stationary problem (63), we have :*

i) $\dfrac{\partial\phi_{loc}^{n+1}}{\partial n}$ *converges to* $\dfrac{\partial\phi}{\partial n}$ *in* $H^{-1/2}(\Gamma_b)$,

ii) ϕ^{n+1} *converges to ϕ in* $H^{1/2}(\Gamma_i)$,

iii) ϕ^{n+1} *converges to ϕ in* $H^1(\Omega)$,

iv) ϕ_{loc}^{n+1} *converges to ϕ in* $H^1(\Omega_{loc})$.

Proof of theorem 5.1:

By the transformation $\phi^{n+1} \to \phi^{n+1} - \phi$ with ϕ the solution of the stationary problem, this problem can be reduced to the case $\phi_\infty = 0$. Multiplying the equation in (64) by $w \in W$, integrating by parts, we obtain:

$$\int_{\Omega_{loc}} \frac{\phi_{loc}^{n+1}}{\tau} w + \int_{\Omega_{loc}} V.\nabla\phi_{loc}^{n+1} w + \nu \int_{\Omega_{loc}} \nabla\phi_{loc}^{n+1} \nabla w$$

$$= \nu \int_{\Gamma_i} \frac{\partial\phi_{loc}^{n+1}}{\partial n} w, \quad \forall w \in W. \tag{66}$$

We now apply the theorem 4.1 and we obtain

$$\begin{aligned} \|\frac{\partial \phi_{loc}^{n+1}}{\partial n}\|_{-1/2,\Gamma_b} \quad \leq \quad & c_1' \sqrt{c_2'/d} \left(1 + \frac{1}{\nu^2}(1 + \|V\|_\infty^2)\right) \\ & \left(1 + \frac{1}{\nu}\|V\|_\infty \sqrt{d/c_2'}\right)^{1/2} \\ & (1 + 1/\tau^2)\exp(-kd^2/36)\|\phi^n\|_{1/2,\Gamma_i}. \end{aligned} \tag{67}$$

On the other hand, multiplying the equation in (65) by $w \in W$ and integrating by parts we obtain the equality

$$\int_\Omega \frac{\phi^{n+1}}{\tau} w + \int_\Omega V.\nabla \phi^{n+1} w + \nu \int_\Omega \nabla \phi^{n+1} \nabla w = \nu \int_{\Gamma_b} \frac{\partial \phi_{loc}^{n+1}}{\partial n} w \tag{68}$$

with $w \in H^1(\Omega)$ and $w = 0$ on Γ_∞. Applying the theorem 3.1 to this problem yields:

$$\begin{aligned} \|\phi^{n+1}\|_{1/2,\Gamma_i} \quad \leq \quad & c_1 \sqrt{c_2/d} \left(1 + \frac{1}{\nu}\|V\|_\infty \sqrt{d/c_2}\right)^{1/2} \\ & \exp(-kd^2/36)\|\partial \phi_{loc}^{n+1}/\partial n\|_{-1/2,\Gamma_b}. \end{aligned} \tag{69}$$

Combinig (67) and (69), we then have:

$$\begin{aligned} \|\partial \phi_{loc}^{n+1}/\partial n\|_{-1/2,\Gamma_b} \quad \leq \quad & c_1' \sqrt{c_2'/d}(1 + \frac{1}{\nu^2}\left(1 + \|V\|_\infty^2)\right) \\ & \left(1 + \frac{1}{\nu}\|V\|_\infty \sqrt{d/c_2'}\right)^{1/2} \\ & c_1 \sqrt{c_2/d} \left(1 + \frac{1}{\nu}\|V\|_\infty \sqrt{d/c_2}\right)^{1/2} \\ & (1 + 1/\tau^2)\exp\left(-k\frac{d^2}{18}\right) \|\partial \phi_{loc}^n/\partial n\|_{-1/2,\Gamma_b}, \end{aligned}$$

with $k = \dfrac{\beta}{\nu\sqrt{\tau}}$. Therefore for τ sufficiently small, the coefficient of reduction will be dominated by the exponential term and will then be strictly less than 1, implying the linear convergence to zero of

$$\|\partial\phi_{loc}^{n+1}/\partial n\|_{-1/2,\Gamma_b}.$$

This corresponds exactly to the statement (i). This statement combined with (69) leads to the convergence of ϕ^{n+1} to 0 in $H^{1/2}(\Gamma_i)$. Applying (14) with $g = \partial\phi_{loc}^{n+1}/\partial n$, we have in addition

$$\|\phi^{n+1}\|_{1,\Omega} \leq \frac{c_o}{\nu}\|\partial\phi_{loc}^{n+1}/\partial n\|_{-1/2,\Gamma_b},$$

and therefore $\|\phi^{n+1}\|_{1,\Omega}$ converges to zero at the speed of $\|\partial\phi_{loc}^{n+1}/\partial n\|_{-1/2,\Gamma_b}$. Applying now (45) with $h = \phi^n$, we also have

$$\|\phi_{loc}^{n+1}\|_{1,\Omega_{loc}} \leq 2(1+1/\tau^2)^{1/2}\left(1+\frac{1}{\nu^2}(1+\|V\|_\infty^2)\right)^{1/2}\|\phi^n\|_{1/2,\Gamma_i}.$$

And then $\|\phi^{n+1}\|_{1,\Omega}$ also converges to zero at the speed of $\|\phi^n\|_{1/2,\Gamma_i}$.

■

5.1 Convergence of a fixed point method for the implicit time marching algorithm

The implicit time marching algorithm of [15] couples the global and the local problem. To uncouple them, it is advisable to use the fixed point algorithm below :

- set $\phi_{loc,0}^o = \psi_{ol}$ and $\phi^o = \psi_o$,
- then for $k \geq 0, \phi_{k|_{\Gamma_i}}^{n+1}$ being known,

  solve

$$\left\{\begin{aligned} \frac{\phi_{loc,k+1}^{n+1}-\phi_{loc}^n}{\Delta t} + \mathrm{div}(v\phi_{loc,k+1}^{n+1}) - \nu\Delta\phi_{loc,k+1}^{n+1} &= 0 \quad \text{in} \quad \Omega_{loc}, \\ \phi_{loc,k+1}^{n+1} &= \phi_k^{n+1} \quad \text{on} \quad \Gamma_i, \\ \phi_{loc,k+1}^{n+1} &= 0 \quad \text{on} \quad \Gamma_b, \end{aligned}\right. \tag{70}$$

$$\left\{\begin{array}{rcl}\dfrac{\phi_{k+1}^{n+1}-\phi^n}{\Delta t}+\mathrm{div}(v\phi_{k+1}^{n+1})-\nu\Delta\phi_{k+1}^{n+1} &=& 0 \quad \text{in} \quad \Omega,\\ \phi_{k+1}^{n+1} &=& \phi_\infty \quad \text{on} \quad \Gamma_\infty,\\ \nu\partial\phi_{k+1}^{n+1}/\partial n &=& \nu\partial\phi_{loc,k+1}^{n+1}/\partial n \quad \text{on} \quad \Gamma_b.\end{array}\right. \tag{71}$$

We will study now the algorithm (70)-(71). By setting

$$\psi_{loc,k,q}=\phi_{loc,k+1}^{n+1}-\phi_{loc,q+1}^{n+1}, \tag{72}$$

$$\psi_{k,q}=(\phi_k^{n+1}-\phi_q^{n+1}), \tag{73}$$

we have that $\psi_{loc,k,q}$ and $\psi_{k,q}$ verify the following equations :

$$\left\{\begin{array}{rcl}\psi_{loc,k,q}/\Delta t+\mathrm{div}(v\psi_{loc,k,q})-\nu\Delta\psi_{loc,k,q} &=& 0 \quad \text{in} \quad \Omega_{loc},\\ \psi_{loc,k,q} &=& \psi_{k-1,q-1} \quad \text{on} \quad \Gamma_i,\\ \psi_{loc,k,q} &=& 0 \quad \text{on} \quad \Gamma_b,\end{array}\right. \tag{74}$$

$$\left\{\begin{array}{rcl}\psi_{k,q}/\Delta t+\mathrm{div}(v\psi_{k,q})-\nu\Delta\psi_{k,q} &=& 0 \quad \text{in} \quad \Omega,\\ \psi_{k,q} &=& 0 \quad \text{on} \quad \Gamma_\infty,\\ \nu\dfrac{\partial\psi_{k,q}}{\partial n} &=& \nu\dfrac{\partial\psi_{loc,k,q}}{\partial n} \quad \text{on} \quad \Gamma_b.\end{array}\right. \tag{75}$$

If Δt is sufficiently small, we can apply the analysis of the previous section to this algorithm and we conclude that $\psi_{k,q}$ and $\psi_{loc,k,q}$ converge linearly to zero. Hence the sequences ϕ_k^{n+1} and $\phi_{loc,k}^{n+1}$ are Cauchy sequences, which converge linearly to the unique solutions ϕ^{n+1} and ϕ_{loc}^{n+1} of the implicit scheme. This guarantees the convergence of the above fixed point algorithm.

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Appendix

In this appendix we shall give the proof of the theorem 2.1 of section 2. This proof relies on the notion of a contact set. If u is a continuous arbitrary function on Ω, the upper contact set, denoted Γ^+ or Γ_u^+, is the subset of Ω, defined by

$$\Gamma^+ = \{y \in \Omega, \exists p(y) \in \mathbb{R}^n \text{such that } u(x) \leq u(y) + p \cdot (x-y) \; \forall x \in \Omega \; \}. \tag{76}$$

We see that u is a concave function on Ω iff $\Gamma^+ = \Omega$. When $u \in C^1(\Omega)$ we must have $p = Du(y)$ in the relation (76). In addition, when $u \in C^2(\Omega)$, the Hessian matrix $D^2 u = [D_{ij} u]$ is negative on Γ^+. In general, Γ^+ is closed in Ω.

If u is a continuous arbitrary function on Ω, we define the "normal mapping" $\chi(y) = \chi_u(y)$ at point $y \in \Omega$ by

$$\chi(y) = \{p \in \mathbb{R}^n, u(x) \leq u(y) + p.(x-y) \; \forall x \in \Omega\}. \tag{77}$$

We can see that $\chi(y)$ is non empty iff $y \in \Gamma^+$. In addition when $u \in C^1(\Omega)$, we have $\chi(y) = Du(y)$ on Γ^+; in other words χ is the gradient field of u on Γ^+.

As a particular case of the Bakelman-Alexandrov ([8] and [9]) maximum principle, we have under the above notation.

Lemma .1 *For $u \in C^2(\Omega) \cap C^o(\bar{\Omega})$, we have :*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{d}{n w_n^{1/n}} \|a^{ij} D_{ij} u / \mathcal{D}^*\|_{n,\Gamma^+}$$

with d the diameter of Ω and w_n the volume of a unit sphere in R^n.

For further details see [12].

We now proceed to the proof of Theorem 6.1, by following the steps of [12]. We take $\hat{B} = B_1(0)$ and the general case will be deduced by considering the coordinate transform, $x \to \hat{x} = (x-y)/2R$.

We will begin, in first step, by showing this result for $u \in C^2(\Omega) \cap W^{2,n}(\Omega)$ and then in a second step we will deduce the result for $u \in W^{2,n}(\Omega)$.

Step 1:

We suppose that $u \in C^2(\Omega) \cap W^{2,n}(\Omega)$. For $\beta \geq 1$, we consider the cut off function η defined by

$$\eta(\hat{x}) = (1 - |\hat{x}|^2)^\beta.$$

By differentiation, we obtain

$$\hat{D}_i \eta = -2\beta \hat{x}_i (1 - |\hat{x}|^2)^{\beta - 1},$$

$$\hat{D}_{ij} \eta = -2\beta \delta_{ij} (1 - |\hat{x}|^2)^{\beta - 1} + 4\beta(\beta - 1)\hat{x}_i \hat{x}_j (1 - |\hat{x}|^2)^{\beta - 2}.$$

By setting

$$v = \eta u,$$

we then obtain

$$\begin{aligned} \hat{a}^{ij} \hat{D}_{ij} v &= \eta \hat{a}^{ij} \hat{D}_{ij} u + 2\hat{a}^{ij} \hat{D}_i \eta \hat{D}_j u + u \hat{a}^{ij} \hat{D}_{ij} \eta \\ &\geq \eta(\hat{f} - \hat{b}^i \hat{D}_i u - \hat{c} u) + 2\hat{a}^{ij} \hat{D}_i \eta \hat{D}_j u + u \hat{a}^{ij} \hat{D}_{ij} \eta. \end{aligned}$$

Let $\Gamma^+ = \Gamma_v^+$ be the upper contact set v, in the sphere $\hat{B}$; we have :

$$u > 0 \text{ on } \Gamma^+.$$

If $x \in \partial \hat{B}$ such that $p.(x - y) < 0$ we indeed have $v(x) = 0$. Consequently

$$v(y) + p.(x - y) \geq v(x) = 0.$$

Moreover, using the concavity of v on Γ^+, we can estimate the following quantity :

$$|\hat{D}u| = (1/\eta)|\hat{D}v - u\hat{D}\eta|.$$

Indeed,

$$\begin{aligned} \left|\hat{D}u\right| &\leq (1/\eta)(\left|\hat{D}v\right| + u\left|\hat{D}\eta\right|) \\ &\leq (1/\eta)(\frac{v}{1 - |\hat{x}|} + u|\hat{D}\eta|) \\ &\leq 2(1 + \beta)\eta^{-1/\beta} u. \end{aligned}$$

In that way, we have on Γ^+ the following inequality :

$$-\hat{a}^{ij}\hat{D}_{ij}v \leq \{(16\beta^2 + 2\eta\beta)\hat{\Lambda}\eta^{-2/\beta} + 2\beta|\hat{b}|\eta^{-1/\beta} + \hat{c}\}v + \eta|\hat{f}|.$$

Since $\hat{c} \leq 0$, we deduce the inequality

$$\begin{aligned} -\hat{a}^{ij}\hat{D}_{ij}v &\leq \{(16\beta^2 + 2\eta\beta)\hat{\Lambda}\eta^{-2/\beta} + \\ &\quad 2\beta|\hat{b}|\eta^{-1/\beta}\}v + \eta|\hat{f}| \\ &\leq c_1\hat{\lambda}\eta^{-2/\beta}v + |\hat{f}|, \end{aligned} \tag{78}$$

with $c_1 = c(n, \beta, \gamma, \hat{\delta})$ independent of $\hat{c}$.

Consequently, by applying Lemma 6.1 on $\hat{B}$, we obtain, for $\beta \geq 2$:

$$\sup_{\hat{B}} v \leq (\frac{\hat{d}}{nw_n^{1/n}})(\frac{1}{\hat{\mathcal{D}}^*})\|c_1\hat{\lambda}\eta^{-2/\beta}v + |\hat{f}|\|_{n,\hat{B}}.$$

By using the relation (2), it comes

$$\begin{aligned} \sup_{\hat{B}} v &\leq (\frac{\hat{d}}{nw_n^{1/n}})c_1\|\eta^{-2/\beta}v\|_{n,\hat{B}} + (\frac{\hat{d}}{nw_n^{1/n}})(\frac{1}{\hat{\lambda}})\|\hat{f}\|_{n,\hat{B}} \\ &\leq c_1\hat{d}(\|\eta^{-2/\beta}v\|_{n,\hat{B}} + (1/\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}) \\ &\leq c_1\hat{d}(\|\eta^{-2/\beta}v^+\|_{n,\hat{B}} + (1/\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}) \\ &\leq c_1\hat{d}((supv^+)^{1-2/\beta}\|(u^+)^{2/\beta}\|_{n,\hat{B}} + (1/\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}), \end{aligned}$$

where c_1 is a constant depending only on n, β, γ and $\hat{\delta}$. Here, $\hat{d}$ is the diameter of $\hat{B}(\hat{d} = 2)$.

By using the Young inequality under the form

$$ab \leq \varepsilon a^q + \varepsilon^{-r/q}b^r$$

for $q = (1 - 2/\beta)^{-1}$ and $r = \beta/2$, we have

$$(supv^+)^{1-2/\beta}\|(u^+)^{2/\beta}\|_{n,\hat{B}} \leq \varepsilon supv^+ + \varepsilon^{1-\beta/2}\|(u^+)^{2/\beta}\|_{n,\hat{B}}^{\beta/2}, \quad \forall\varepsilon > 0.$$

By taking $\varepsilon = \dfrac{1}{2c_1\hat{d}}$ and plugging in our inequality on v, we obtain :

$$\sup_{\hat{B}} v \leq (1/2)\sup v^+ + (1/2)^{1-\beta/2}(c_1\hat{d})^{\beta/2}\|(u^+)^{2/\beta}\|_{n,\hat{B}}^{\beta/2} + (c_1\hat{d}/\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}. \tag{79}$$

We want to prove the theorem for all $p > 0$. We will treat separately the cases $p \leq n$ and $p > n$.

If $p \leq n$, we set $\beta = 2n/p$. In this case we have

$$\|(u^+)^{2/\beta}\|_{n,\hat{B}}^{\beta/2} = \|(u^+)\|_{p,\hat{B}}.$$

Plugging this in our inequality on v, we obtain :

$$(1/2)\sup_{\hat{B}} v \leq (1/2)^{1-\beta/2}(c_1\hat{d})^{\beta/2}\|(u^+)\|_{p,\hat{B}} + (c_1\hat{d}/\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}.$$

Consequently, we obtain the following inequality ;

$$\sup_{\hat{B}} v \leq c_2\{(\int_{\hat{B}}(u^+)^p)^{1/p} + (\hat{d}/2\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}\}.$$

On the sphere $B_{1/2}(0)$, the cut off function satisfies

$$1/\eta \leq (1/2)^{\beta}.$$

It follows, then

$$\begin{aligned}\sup_{B_{1/2}(0)} u &\leq \sup_{B_{1/2}(0)} (v/\eta) \\ &\leq 2^{\beta}\sup_{\hat{B}} v.\end{aligned}$$

Finally we end up at the desired estimate

$$\sup_{B_{1/2}(0)} u \leq c_3\{(\int_{\hat{B}}(u^+)^p)^{1/p} + (\hat{d}/2\hat{\lambda})\|\hat{f}\|_{n,\hat{B}}\}.$$

for u in $W^{2,n}(\Omega)\cap C^2(\bar{\Omega})$. The constant c_3 above depend only on n, β, γ and $\hat{\delta}$, but is independent of $\hat{c}$.

On the other hand if $p > n$, we have :

$$2n/\beta < p, \ \forall \beta \geq 2.$$

Then, it follows (by assuming $\beta \geq 2$)

$$|\hat{B}|^{-1/(2n/\beta)} \|(u^+)\|_{2n/\beta,\hat{B}} \leq |\hat{B}|^{-1/p} \|u^+\|_{p,\hat{B}}.$$

But

$$\|u^+\|_{2n/\beta} = \|(u^+)^{2/\beta}\|_{n,\hat{B}}^{\beta/2},$$

and therefore, by processing as before, we obtain the desired estimate

$$\sup_{B_{1/2}(0)} u \leq c_4 \{ (\int_{\hat{B}} (u^+)^p)^{1/p} + (\hat{d}/2\hat{\lambda}) \|\hat{f}\|_{n,\hat{B}} \}$$

for u in $W^{2,n}(\Omega) \cap C^2(\bar{\Omega})$. The constant c_4 above depends only on n, β, γ and $\hat{\delta}$, but is independent of $\hat{c}$.

Transformation $\hat{x} \to x$.

By construction, $\hat{D}_{ij} = R^{-2} D_{ij}$, thus $\hat{\lambda} = R^{-2}\lambda$ and $\hat{\delta} = \delta R^2$. In addition, we have $|B| = w_n (2R)^n$ and $|g|_{p,\hat{B}} = R^{-n/p} |g|_{p,B}$.

Written in term of x, the last inequality becomes

$$\sup_{B_{R(y)}} u \leq c_4 \{ (\frac{2^n w_n}{|B|} \int_B (u^+)^p dx)^{1/p} + (\frac{2 w_n^{1/n} R}{\lambda}) \|f\|_{n,B} \},$$

with c_4 a function of $n, \gamma, \hat{\delta} = \delta R^2$ and p. This is the desired estimate for $u \in W^{2,n}(\Omega) \cap C^o(\bar{\Omega})$.

Step 2:

Now, let $u \in W^{2,n}(\Omega)$. By density argument, let u_m be a sequence of functions of $C^2(\bar{B})$, converging towards u in $W^{2,n}(B)$. The injection of $W^{2,n}(B)$ in $C^o(B)$ is continuous, consequently u_m converges uniformly towards u in B. We have

$$\begin{aligned} L u_m &= L(u_m - u) + Lu \\ &\geq f + L(u_m - u). \end{aligned}$$

By setting, $f_m = L(u_m - u)$, we observe by construction that f_m converges towards 0 in $L^n(\Omega)$. As $u_m \in W^{2,n}(\Omega) \cap C^2(\Omega)$ and $\tilde{f}_m = f + f_m$ is in $L^n(\Omega)$, the estimate (3) is valid also for u_m, so that we have

$$\sup_{B_R(y)} u_m \leq \text{const}\{(\frac{1}{|B|}\int_B (u_m^+)^p)^{1/p} + \frac{R}{\lambda}\|\tilde{f}\|_{n,B}\}. \tag{80}$$

Using previous results and taking the limit, we have :

$$\sup_{B_R(y)} u \leq \text{cte}\{(\frac{1}{|B|}\int_B (u^+)^p)^{1/p} + \frac{R}{\lambda}\|f\|_{n,B}\}.$$

■

Observe also that by replacing u by $-u$, the theorem can be extended easily to the case of supersolutions and solutions of the equation :

$$Lu = f.$$