Fluctuating dilatation rate as an acoustic source

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Abstract [1]

Ribner's (1962) dilatational acoustic theory is revisited. A rigorous connection between the fluctuating dilatation rate and the acoustic source field is established; this vindicates Ribner's heuristic contention while indicating additional acoustic source terms in his dilatational acoustic theory. It is also shown that Ribner's acoustic source term is quadrupole. Interesting consequences of the dilatational point of view are indicated. It is found that in the region of vortical fluid motion the dilatation scales as the square of the turbulent Mach number, M_t^2, and has little to do with the acoustic field; its time rate of change, however, is a portion of the sound generation mechanism. Away from the vortical region the fluid dilatation is an acoustic variable and scales as M_t^4. The mathematical link established is useful to interpreting direct numerical simulation of aeroacoustical flows in which the dilatation is computed.

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1. Introduction

In the nascent field of direct computation of aeroacoustical flows, the dilatational fields are sometimes investigated, see for example, Colonius, Lele, and Moin [1] or Mitchell, Lele, and Moin [2]. One has, in general, an intuitive appreciation of the fact that sound and dilatation are linked. This link has not been *rigorously* confirmed for the aerodynamic source problem. Links have been made to fluctuating Reynolds stresses, Lighthill [3], Phillips [4], the fluctuating vorticity, Powell [5], Howe [6], the pseudo-pressure, Ribner [7], the fluctuating triple velocity gradient, Lilley (1974) (more easily found in Goldstein [10]) and the fluctuating enthalpy Howe [6]. This article also provides and clarifies some mathematical details missing in [7]'s heuristic analysis. In addition to formally establishing the link between the dilatation rate and sound source some of Ribner's contentions are vindicated. Useful DNS consequences of the dilatational point of view are indicated. As the dilatational approach to an aeroacoustic theory appears to have generated some controversy some three decades ago, it seems necessary to make clear at the outset that the object of this article is not related to the use of the dilatational theory as an acoustic analogy.

In the present era, some useful items related to Ribner's [7] dilatational theory suggest a reappraisal of Ribner's ideas. Ribner [7] related the acoustic source to the second time derivative of the incompressible pressure. Ribner [7] gives physical arguments relating his sound source to the fluctuating dilatation. His arguments have not, however, been given formal mathematical expression or justification. This article formally establishes Ribner's [7] heuristic assertion: that sound generation is related to the *time rate of change of dilatation* [2]. The derivation of this mathematical connection and related consequences as well as some useful aspects of the dilatational point of view are the subjects of this article. In addition to exploring and clarifying Ribner's [7] ideas some useful and relevant consequences for DNS and LES are obtained. In focussing on Ribner's analysis, the discussion is limited to compact flows with negligible nonisentropic or diffusive effects. Implicit in the development is the limitation to the class of flows for which the Lighthill form of the acoustic analogy is useful. Many of the effects accounted for in more advanced acoustic theories, such as those of Lilley or Phillips, are not addressed in this article.

In summary: 1) a formal link between the sound source and the dilatation is derived; 2) Ribner's source term is shown to be quadrupole; 3) additional source terms in Ribner's dilatational theory are determined; and, 4) the potential of the time rate of change of the dilatation as a diagnostic tool is indicated.

[2] For this article the quantity $u_{i,i}$ will be called the dilatation (it does in fact have units of a rate).

2. Some preliminaries and the Ribner analysis

A few introductory comments and an informal outline of some useful physical ideas, following Ribner [7], are summarized. In this section Ribner's [7] mathematical development is followed; this is in order to highlight its differences with a more mathematical analysis.

For the present class of flows the term $(p - c_\infty^2 \rho)_{,tt}$, following [7], is neglected and the pressure fluctuations in a fluid satisfy the following Lighthill equation,

$$c_\infty^{-2} p_{,tt} - p_{,jj} = (\rho u_i u_j)_{,ij} \tag{1}$$

where p represents the deviations of the fluid pressure from a static reference value. In medium of uniform mean density ρ_∞, it is customary to approximate $\rho u_i u_j = \rho_\infty v_i v_j$ since higher order terms scale as the square of the fluctuating Mach number. Here v_i represents the solenoidal velocity. Ribner [7] assumes at the outset the validity of this approximation. Ribner distinguishes two pressures, an "acoustic" pressure which propagates and a "pseudo-pressure" [3] associated with the convective motions of the fluid. The pressure field in the Lighthill equation includes both acoustic and convective pressures. The fluid pressure, following [7], is partitioned into convective and propagating parts $p = p_s + p_a$ where p_s satisfies

$$-\nabla^2 p_s = \rho_\infty (v_i v_j)_{,ij} \,. \tag{2}$$

The subscript s is used to denote the incompressible or pseudo-pressure associated with the solenoidal velocity field. The acoustic pressure, p_a, following Ribner satisfies

$$p_{a,tt} - c_\infty^2 \nabla^2 p_a = -p_{s,tt} \tag{3}$$

– the sound source has been related to the incompressible pressure which satisfies the Poisson equation whose solution is given by convolution. Ribner [7] then goes on to argue, without giving proof, that $\ddot{p}_s$ (the overdot signifies a time derivative) is related to the dilatation.

The following section establishes a rigorous connection of Ribner's contention of a relation between the pseudo-pressure sound source and the dilatation. Some subtleties regarding Ribner's very useful decomposition $p = p_s + p_a$ are described. It is seen that the decomposition retains terms that are of the same order in M_t^2 as terms discarded in the truncation $\rho u_i u_j = \rho_\infty v_i v_j$.

3. A perturbation analysis leading to Ribner's acoustic analogy

The following analysis is understood to address high Reynolds number, weakly compressible (low fluctuating Mach number), turbulent compact flows with constant mean density – the class of flows

[3] The term "pseudo-pressure" coined by Blokhintsev (1956) as quoted in [7] is adopted.

relevant to subsonic aeroacoustics for which the energy equation implies $p \sim \rho^\gamma$. A low fluctuating Mach number perturbation analysis of the compressible Navier-Stokes equations,

$$
\begin{aligned}
\rho,_t + u_k \rho,_k &= -\rho u_{k},_{k} && (4)\\
\rho u_{i},_{t} + \rho u_k u_{i},_{k} + p,_i &= 0 && (5)\\
p/p_\infty &= (\rho/\rho_\infty)^\gamma && (6)\\
\rho,_{tt} - p,_{jj} &= (\rho u_i u_j),_{ij} && (7)
\end{aligned}
$$

is required. Perturbing about a reference state, (p_∞, ρ_∞), the nondimensional forms of the pressure and density are taken as $p = p_\infty(1 + p')$, $\rho = \rho_\infty(1 + \rho')$. The independent variables are rescaled with the energy containing length and time scales of the turbulent flow field: ℓ/u_c and ℓ. Dropping primes the compressible Navier-Stokes equations become

$$
\begin{aligned}
\rho,_t + u_k \rho,_k &= -(1+\rho) u_{k},_{k} && (8)\\
(1+\rho) u_{i},_{t} + (1+\rho) u_p u_{i},_{p} + \epsilon^{-2}\, p,_i &= 0 && (9)\\
p - \gamma\rho &= 1/2\, \gamma(\gamma-1)\rho^2 && (10)\\
\rho,_{tt} - \epsilon^{-2} p,_{jj} &= [(1+\rho) u_i u_j],_{ij}\,. && (11)
\end{aligned}
$$

A wave equation related to the last equation, in nondimensional units, is

$$
\gamma^{-1} p,_{tt} - \epsilon^{-2} p,_{jj} = [(1+\rho) u_i u_j],_{ij} + [\gamma^{-1} p - \rho],_{tt}\,. \qquad (12)
$$

In these equations $\epsilon^2 = \gamma M_t^2 = \gamma u_c^2/c_\infty^2$ and $c_\infty^2 = \gamma p_\infty/\rho_\infty$. The characteristic fluctuating velocity might be identified with the energy of the turbulent fluctuations: $u_c^2 \sim \frac{2}{3}k = \frac{1}{3} < u_j u_j >$. The conventional definition of the Mach number (in contradistinction to the larger fluctuating Mach number used in DNS of compressible turbulence) is used. Expansions of the form $p = \epsilon^2\, [\, p_1 + \epsilon^2 p_2 + ...]$, $\rho = \epsilon^2\, [\, \rho_1 + \epsilon^2 \rho_2 + ...]$, $u_i = v_i + \epsilon^2\, [\, w_i + \epsilon^2 w_{2i} + ...]$, are chosen. Substituting the expansions into the equations produces, to the lowest order, the incompressible form of the equations,

$$
\begin{aligned}
v_{i},_{t} + v_p v_{i},_{p} + p_1,_i &= 0 && (13)\\
v_{i},_{i} &= 0 && (14)\\
\nabla^2 p_1 &= -(v_i v_j),_{ij} && (15)\\
p_1 &= \gamma \rho_1. && (16)
\end{aligned}
$$

Note that variables are now dimensionless. The lowest order form of the equations of state indicates that the pressure is the solenoidal or pseudo-pressure [4]. Additional details can be found in Ristorcelli [13].

[4] In dimensional form this is statement that $\rho_1 = c_\infty^{-2} p_s/\rho_\infty$.

The inner form of the wave equation is, to order M_t^2,

$$-\nabla^2 p_2 \ = \ [(1+\rho_1)u_i u_j \ - \ v_i v_j]_{,ij} \ - \gamma^{-1} p_{1,tt} \,. \tag{17}$$

Note that for brevity and clarity, the quantity $u_i u_j$ has not been expressed in the pertubation variables. The term in square brackets is more physically meaningful if left as is: it is the difference between the compressible and incompressible fluctuating Reynolds stresses. The continuity equation, at next order, is $\rho_{1,t} + v_k \rho_{1,k} = -w_{k,k}$. Using the equation of state $p_1 = \gamma \rho_1$, the first order continuity equation yields a *diagnostic* relationship for the leading order dilatation, $d = w_{k,k}$,

$$-\gamma d \ = \ p_{1,t} \ + v_k p_{1,k} \,. \tag{18}$$

Ribner's [7] interpretation of the second time derivative of the pseudo-pressure as dilatation rate is now formally established: $\ddot{p}_1 = -\gamma \dot{d} - (v_k p_k)_{,t}$ and p_2 satisfies, to $\mathcal{O}(M_t^2)$,

$$- \nabla^2 p_2 = \ \dot{d} + \ [(1+\rho_1)u_i u_j \ - \ v_i v_j]_{,ij} \ + \gamma^{-1}(v_k p_1)_{,kt} \,. \tag{19}$$

Lighthill's equation, $c_\infty^{-2} p_{,tt} \ - p_{,jj} \ = \ (\rho u_i u_j)_{,ij}$, can be reconstituted in dimensional terms (without implied length and time scales). Using Ribner's decomposition $p = p_s + p_a$ where the *dimensional* quantities $[p_s, p_a]$ corresponding to the nondimensional $[p_1, p_2]$ produces, to $\mathcal{O}(M_t^2)$,

$$c_\infty^{-2} p_{a,tt} \ - \nabla^2 p_a = \ \gamma c_\infty^{-2} p_\infty \dot{d} + \ \rho_\infty[(1+\rho')u_i u_j \ - \ v_i v_j]_{,ij} \ + \gamma^{-1} c_\infty^{-2}(v_k p_s)_{,kt} \,. \tag{20}$$

The divergence of the pressure flux is an octupole (see Appendix) and of higher order in M_t than the dilatation rate. The wave equation in Ribner's dilatational acoustic theory is, consequently,

$$c_\infty^{-2} p_{a,tt} \ - \nabla^2 p_a = \gamma c_\infty^{-2} p_\infty \dot{d} + \ \rho_\infty[(1+\rho')u_i u_j \ - \ v_i v_j]_{,ij} \,. \tag{21}$$

The dilatation is seen to be *one* portion of the acoustic source; additional quadrupoles (in the square brackets) arise from the compressible nature of the Reynolds stresses. These terms do not appear in the Ribner analysis as $\rho u_i u_j \sim \rho_\infty v_i v_j$ was assumed at outset - a truncation of terms of the same order as $\dot{d}$. (The source term can be expanded $[(1+\rho')u_i u_j \ - \ v_i v_j] \ = v_i w_j + v_j w_i + \rho' v_i v_j + h.o.t.$)

It should be noted out that Crow's [9] analysis of Ribner's theory, in which it is shown that Ribner's source term $\int \ddot{p}_s d\mathbf{y} = 0$, would have been different if the retarded time had been included. Retention of the full retarded time would have given a non-zero integral of Ribner's source term invalidating one of Crow's conclusion regarding Ribner's theory. Ffowcs Williams [12] indicates that $p_s \sim x^{-3}$ and the integral is in fact weakly (logarithmically) divergent requiring very careful source term modeling and integration, if it is to be used as an acoustic analogy. In contradistinction, the source term in the Lighhill analogy scales as the double divergence of $v_i v_j \sim x^{-6}$.

While the expression $c_\infty^{-2} p_a,_{tt} \; - \; \nabla^2 p_a \; = \; \gamma c_\infty^{-2} p_\infty \dot{d} + ...$ identifies a new quantity, $\dot{d}$, it does not contain any new physics. The quadrupole nature of the sound source field is retained as can be seen from the following application of the convolution algebra: The solution for the solenoidal-pressure is

$$p_s(x) = [v_i v_j(x')],_{i'j'} \otimes G_o(x - x') = (v_i v_j) \otimes G_o,_{i'j'} = (v_i v_j) \otimes G_o,_{ij} = [(v_i v_j) \otimes G_o],_{ij} \, . \tag{22}$$

G_o is the three-dimensional Greens function for the Laplacian, $G_o = -\frac{1}{4\pi}|x - x'|^{-1}$. The dilatation rate is related to the second time derivative of the incompressible pressure and the source term is re-expressed as

$$\dot{d} \;\; \sim \;\; \ddot{p}_s \;\; \sim \; [(v_i v_j),_{tt} \otimes G_o],_{ij} \, . \tag{23}$$

In less abstract terms

$$\ddot{p}_s \; = \rho_\infty [\; \frac{1}{4\pi} \int (v_i v_j),_{tt} \; \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \;],_{ij} = S_{ij},_{ij} \tag{24}$$

The source term is a double divergence and the quadrupole nature of the source term is retained. It appears that $\dot{d}$ (and, of course, Ribner's original $\ddot{p}_s$) are to be understood as a quadrupole.

In summary: In this revisit to Ribner's [7] theory 1) the source term $\ddot{p}_s$, has been formally linked to the dilatation; 2) the source term in his theory [7] has been shown to be a *portion* of the sound source; and 3) the source term has been shown to be a quadrupole as is consistent with the Lighthill formalism that underlies Ribner's procedure.

As has been alluded to in the literature, [12] [8], the dilational point of view, despite its mathematically exact derivation, does not appear to be practical with the assumptions that are required in an acoustic analogy. The acoustic analogy formalism requires a model for the source term $\dot{d}$. The dilatation would be related to the solenoidal motions; this is done by definition as $\dot{d}$ would be given in terms of $\ddot{p}_s$. However, a convolution must be taken and the procedure loses compactness and direct relation between sound source and a local (compact) fluid mechanical mechanism is lost; the integral is over a region larger than that which contains the vorticity (Howe [6], Ffowcs Williams [12], Crighton [8]), and integrations must be done very precisely. This is consistent with the scalings $\ddot{p}_s \sim x^{-3}$, $v_i v_j \sim x^{-6}$. The fact that this is not an easily applicable result in the context of an acoustic analogy is not relevant; our purpose is a more systematic clarification of the Ribner's ideas and a presentation of consequences of the dilatational point of view that bear relevance to the DNS of aeroacoustical flows in fundamental studies such as [1], [2].

4. Some consequences in the context of compressible DNS

The dilatation indicated by the diagnostic relationship derived above is the dilatational field associated with the incompressible pressure, p_s. The dilatational field computed in the DNS of a sound generating flow will contain a contribution from the pseudo-sound pressure, p_s, and a higher order contribution from the acoustic pressure, p_a. This can be shown from the next higher order expansion of the continuity equation. In the source field the dilatation associated with the incompressible pressure ("pseudo-dilatation") using the continuity equation, scales as

$$d_s \sim \frac{1}{\rho_\infty}\frac{D}{Dt}\rho \sim \frac{1}{\rho_\infty c_\infty^2}\frac{D}{Dt}p_s \sim M_t^2\frac{u_c}{\ell} \tag{25}$$

since $p_s \sim \rho_\infty u_c^2$. Although d_s is a measure of the compressibility of the flow, it is not to be interpreted as an acoustic variable. In the sourceless region, $\Box^2 p_a = 0$, and the usual linear scalings $p_a = \rho_a c_\infty^2 = \rho_\infty w$ are relevant; thus $w = \frac{\rho_a}{\rho_\infty}c$. For quadrupole radiation the Lighthill analogy, [3], [8], [6], gives $\frac{\rho_a}{\rho_\infty} \sim \frac{\ell}{|x|}M_t^4$, the dilatation associated with the acoustic field scales as

$$d_a \sim \frac{w}{\lambda} \sim \frac{\rho_a}{\rho_\infty}\frac{c}{\lambda} \sim \frac{\rho_a}{\rho_\infty}\frac{u_c}{\ell} \sim M_t^4\frac{u_c}{\ell} \tag{26}$$

since $u_c/\ell \sim c/\lambda$. Thus $d_a \sim M_t^2 d_s$. The acoustic dilatation, in the acoustic regime is, of course, related to the pressure in the usual way, $d_a \sim \mathring{p}_a$ and satisfies $\Box^2 d_a = 0$ as does, of course, $\dot{d}_a$. The dimensional set $[d_s, d_a]$ correspond to the nondimensional set $[d_1, d_2]$, (in our nomenclature the subscript 1 on the leading order term has been dropped).

In fundamental studies involving the compressible simulation of aeroacoustical flows, [1] or [2], both d_a and d_s are calculated. In the source region d_s will dominate and the dilatation has little to do with the acoustic field *per se*; it is, however, the source of the acoustic field. This is subtly acknowledged in [1] in which the near-field dilatation is not shown – see their Figures 5, 6 and 10. The near-field acoustic dilatation could in principle be plotted after subtracting d_s from the computational data. Far enough away from the source region, $|x| > \lambda$, the acoustic portion of the dilatation will of course dominate, $d_a > d_s$ as the dilatation associated with the solenoidal velocity field through the pressure, d_s, which decays since $\dot{d} \sim |p_s| \sim \frac{1}{|x|^3}$.

5. Conclusions

The intention of the present article has been to explore and clarify several issues relating to Ribner's [7] dilatational theory. Given the controversy Ribner's ideas caused some thirty five years ago, the issue appeared to warrant attention that appears to be missing in the published literature. In vindication of Ribner's contention, the sound source has been formally related to the dilatation - a link never rigorously made in Ribner's [7] treatment. The analysis has also indicated, in the context of the flows of interest in Ribner's formulation, additional sound source terms.

There is relevance, in the present era of computational possibilities, of Ribner's ideas as a diagnostic tool to be used to investigate numerical data from compressible DNS. The dilatational rate is one sound source in the near-field and in the far-field it is one of several adjunct acoustic variables $[\rho_a, \phi, p_a, d_a, \dot{d}_a]$ – any one of which serve to generate the linear sound field. Thus in DNS investigations using dilatational fields, the dilatation rate – $\dot{d}$, not the dilatation – d, appears to be the more potentially useful field for the physical interpretation of computational results. It has properties that relate to both the sound generation and the sound propagation aspects of the aeroacoustical problem – its character being determined by where it is evaluated.

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Appendix: Some details

The pressure flux source term, $(v_k p)_{,kt}$, is now shown to be an octupole. The solution for the pressure is $p_s = [(v_i v_j) \otimes G_o]_{,ij} \; = \; P_{ij}{}_{,ij}$. The pressure flux can be written, using $v_k{}_{,k} = 0$,

$$(v_k p)_{,k} \; = \; v_k \, p_{,k} \; = \; v_k \, P_{ij}{}_{,ijk} \tag{27}$$

and the source term is seen to be a triple divergence and thus a more inefficient octupole. For those not familiar with this style of reasoning – the procedure is standard in aeroacoustics – a few overview comments offered. The three-dimensional Greens function for the wave equation is, in its far-field approximation,

$$G_\infty(x', t'; x, t) \; = \; G_\infty(x - x', t - t') \; = \; \frac{1}{4\pi c^2 |x|} \, \delta(t - t' - |x - x'| c^{-1}) \; = \; \frac{1}{4\pi c^2 |x|} \, G_\delta. \tag{28}$$

For a generic source term $S(x) \; = \; S_{ijk...,ijk...}$ skipping the intermediate steps already given, the solution is

$$p_a(x) \; = \; \frac{1}{4\pi c^2 |x|} S_{ijk...,i'j'k'...} \otimes G_\delta = \; \frac{1}{4\pi c^2 |x|} \; S_{ijk...} \otimes \; G_{\delta,ijk...} \quad . \tag{29}$$

The symmetry in time and space of G_δ allows time derivatives to replace spatial derivatives – each additional spatial derivative contributing a power of the fluctuating Mach number as the symmetry in (x, t) allows $\frac{\partial}{\partial x_i} = -\frac{1}{c}\frac{x_i}{|x|}\frac{\partial}{\partial t}$. Thus $G_\delta{}_{,i} = -\frac{1}{c}\frac{x_i}{|x|} G_\delta{}_{,t}$ and

$$p_a(x) \; \sim \; \frac{1}{4\pi c^2 |x|} \; \frac{x_i}{c|x|} \; \frac{x_j}{c|x|} \; \frac{x_k}{c|x|} ... \, [S_{ijk...,ttt...} \otimes G_\delta]. \tag{30}$$

Hence the pressure flux source term, $P_{ij,ijk}$ is an order higher in the turbulent Mach number than the dilatation rate.