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DYNAMICAL SYSTEM ANALYSIS OF REYNOLDS STRESS CLOSURE EQUATIONS

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Abstract. In this paper, we establish the causality between the model coefficients in the standard pressure-strain correlation model and the predicted equilibrium states for homogeneous turbulence. We accomplish this by performing a comprehensive fixed point analysis of the modeled Reynolds stress and dissipation rate equations. The results from this analysis will be very useful for developing improved pressure-strain correlation models to yield observed equilibrium behavior.

Key words. turbulence modeling, pressure-strain modeling

Subject classification. Fluid Mechanics

1. Introduction. The equilibrium states of benchmark turbulent homogeneous flows have long been used to develop, calibrate and validate pressure-strain correlation models, Speziale *et al* (1991). The benchmark flows have typically been plane shear and strain-rate dominated flows such as plane strain and axisymmetric expansion/contraction. When these pressure-strain models are used to compute slightly more complex flows which contain the effects of both rotation and strain (elliptic flows), the model results are inconsistent with linear stability theory and direct numerical simulation (DNS) data, (Speziale *et al* 1996, Blaisdell and Shariff, 1996). Improved predictive capability of elliptic flows requires a better understanding of the physics of turbulence as well as the dynamics of the model equations. This calls for an intimate understanding of the model equations, especially the causality between the model constants and the predicted long-time behavior. For the special case of shear flow in rotating coordinate frame, the equilibrium state predicted by two-equation turbulence models and linear pressure-strain models have been investigated by Speziale and Mac Giolla Mhuiris (1989, 1990). These studies have lead to models with better predictive capability in those selected flows.

The objective of the present study is to perform dynamical system analysis, i.e., fixed point and bifurcation analyses, of the anisotropy, kinetic energy and dissipation closure equations with aid of representation theory for all elliptic flows (with two-dimensional mean velocity field) to establish for the first time exact (analytical) relationship between turbulence model (pressure-strain correlation and dissipation equation) coefficients and the asymptotic behavior of the equation. The results from our study enables us to classify completely the turbulent asymptotic state predicted by a model as a function of the mean strain and rotation rates. The knowledge of this causality can be used in closure model development to yield the required asymptotic (equilibrium) behavior. Quasilinear pressure-strain model which includes the entire class of Launder, Reece and Rodi (1975), referred to as LRR, models are considered here. This study also sheds light on the strategies that can be employed to develop non-equilibrium algebraic Reynolds stress models starting form the Reynolds stress closure equations.

In this study we will restrict ourselves to turbulence in inertial frames which includes elliptic flows. Turbulence modeling in non-inertial rotating frames will be considered in future work.

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2. Turbulence Closure Equations. In homogeneous turbulence, the exact Reynolds stress transport equation in an arbitrary inertial reference frame is given by

$$\frac{d\overline{u_i u_j}}{dt^*} = P_{ij} - \varepsilon_{ij} + \phi_{ij}. \tag{1}$$

The terms, respectively, are the time rate of change, production (P_{ij}), dissipation (ε_{ij}) and pressure-strain correlation (ϕ_{ij}) of Reynolds stress:

$$\begin{aligned} P_{ij} &= -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}; \\ \varepsilon_{ij} &= 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}; \quad \phi_{ij} = \overline{p(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})}. \end{aligned} \tag{2}$$

The production and dissipation rate of turbulent kinetic energy are, respectively, $P = \frac{1}{2}P_{ii}$ and $\varepsilon = \frac{1}{2}\varepsilon_{ii}$. In high Reynolds number flows, dissipation is generally treated as being isotropic:

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}. \tag{3}$$

Closure models are needed for the pressure-strain correlation (ϕ_{ij}) and dissipation rate (ε).

In this study, we focus on the quasilinear class of pressure-strain correlations models of the general form:

$$\begin{aligned} \phi_{ij} = &-(C_1^0\varepsilon + C_1^1 P)b_{ij} + C_2 K S_{ij} + \\ &C_3 K(b_{ik}S^*_{jk} + b_{jk}S^*_{ik} - \frac{2}{3}b_{mn}S^*_{mn}\delta_{ij}) + \\ &C_4 K(b_{ik}W^*_{jk} + b_{jk}W^*_{ik}), \end{aligned} \tag{4}$$

where the C's are numerical constants and

$$\begin{aligned} S^*_{ij} &= \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}); \quad W^*_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}); \\ b_{ij} &= \frac{\overline{u_i u_j}}{2K} - \frac{1}{3}\delta_{ij}. \end{aligned} \tag{5}$$

We choose this form of the pressure-strain model for two reasons. First, this form of the model permits analytical treatment of the asymptotic behavior. Second, this is the form most frequently used in practical Reynolds stress closure calculations: this form includes all linear-pressure strain models (e.g. LRR model) and some of the non-linear models (such as the quasi-linear SSG model) can also be reduced to this form near equilibrium. For the LRR model, the coefficients are

$$C_1^0 = 3.0;\ C_1^1 = 0.;\ C_2 = 0.8;\ C_3 = 1.75;\ C_4 = 1.31$$

The coefficients for the Gibson and Launder (1978) model are

$$C_1^0 = 3.6;\ C_1^1 = 0.;\ C_2 = 0.8;\ C_3 = 1.2;\ C_4 = 1.2$$

For the quasilinearized SSG model, the coefficients are

$$C_1^0 = 3.4;\ C_1^1 = 1.8;\ C_2 = 0.36;\ C_3 = 1.25;\ C_4 = 0.4$$

The anisotropy evolution equation can be derived from the Reynolds stress equation and the pressure strain correlation model:

$$\begin{aligned} \frac{db_{ij}}{dt^*} = &-b_{ij}(L_1^0\frac{\varepsilon}{K} - L_1^1 b_{mn}S^*_{mn}) + L_2 S^*_{ij} \\ &+L_3(b_{ik}S^*_{jk} + b_{jk}S^*_{ik} - \frac{2}{3}b_{lm}S^*_{lm}\delta_{ij}) \\ &+L_4(b_{ik}W^*_{jk} + b_{jk}W^*_{ik}) \end{aligned} \tag{6}$$

The pressure-strain correlation model coefficients are redefined as:

$$
\begin{aligned}
&L_1^0 \equiv C_1^0 - 2; \;\; L_1^1 \equiv 2C_1^1 + 4; \;\; L_2 \equiv C_2 - \frac{4}{3}; \\
&L_3 \equiv C_3 - 2; \;\; L_4 \equiv C_4 - 2.
\end{aligned} \tag{7}
$$

The turbulent kinetic energy evolves according to

$$
\frac{dK}{dt^*} = P - \varepsilon, \tag{8}
$$

and the modeled evolution equation of dissipation is

$$
\frac{d\varepsilon}{dt^*} = C_{e1}\frac{\varepsilon}{K}P - C_{e2}\frac{\varepsilon^2}{K}. \tag{9}
$$

The model constants C_{e1} and C_{e2} are typically given values of 1.44 and 1.90 respectively.

Equations (6), (8) and (9) constitute the second order closure equations in homogeneous turbulence. These equations can be non-dimensionalized using the norm of the deformation rate tensor:

$$
\eta = S_{ij}^* S_{ij}^* + W_{ij}^* W_{ij}^*, \tag{10}
$$

The non-dimensional quantities are

$$
\begin{aligned}
&S_{ij} = S_{ij}^*/\sqrt{\eta}; \, W_{ij} = W_{ij}^*/\sqrt{\eta}; \\
&dt = \sqrt{\eta} dt^*; \, \omega = \varepsilon/(\sqrt{\eta}K),
\end{aligned} \tag{11}
$$

where ω is the relative strain rate, i.e., the ratio of the turbulence to mean flow strain rates. In the above equations, asterisk is used to represent dimensional quantities and the corresponding non-dimensional quantity is written without the asterisk. In dimensionless time, the anisotropy transport equation is

$$
\begin{aligned}
\frac{db_{ij}}{dt} = &-b_{ij}(L_1^0\omega - L_1^1 b_{mn}S_{mn}) + L_2 S_{ij} \\
&+L_3(S_{ik}b_{kj} + b_{ik}S_{kj} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}) \\
&+L_4(W_{ik}b_{kj} - b_{ik}W_{kj}).
\end{aligned} \tag{12}
$$

Two points are worthy of note here. In dimensionless time the anisotropy evolution is (i) independent of the magnitude of deformation (η) and depends only on the ratio of strain to rotation rate; and (ii) dependent only on the relative strain rate and not individually on kinetic energy and dissipation. The evolution equation of the relative strain rate, ω, is easily obtained from those of the turbulent kinetic energy and dissipation:

$$
\frac{d\omega}{dt} = -2\omega(C_{e1} - 1)b_{mn}S_{mn} - (C_{e2} - 1)\omega^2. \tag{13}
$$

The first term on the right hand side of equation (13) represents the production of the relative strain rate whereas the second terms represents its destruction.

In homogeneous turbulence, the above non-linear dynamical system of equations represents an initial value problem. The Reynolds stress, kinetic energy and dissipation may grow unbounded from their initial values. However, it is known that for some benchmark flows such as (homogeneous) plane shear, plane strain and axisymmetric expansion/contraction the normalized turbulence parameters b_{ij} and ω evolve from their specified initial conditions according to the equations and asymptote to finite-valued fixed points (equilibrium turbulence), provided such a state exists. The fixed point or the equilibrium state of turbulence is described by

$$
\frac{db_{ij}}{dt} = 0; \;\; and \;\; \frac{d\omega}{dt} = 0. \tag{14}
$$

2.1. Representation theory. We are interested in the asymptotic behavior, long after the influence of the initial conditions has diminished. At this stage of flow evolution, it can be argued that the Reynolds stress can be a tensorial function of only the mean strain and rotation rates. Representation theory can then be invoked to determine the most general tensor function that can be constructed with the strain and rotation rates. The most general, physically permissible tensor representation for the Reynolds stress anisotropy in terms of the strain and rotation rates in the case of two-dimensional mean flow is given by (Girimaji 1996a)

$$\begin{aligned} b_{ij} = G_1 S_{ij} + G_2(S_{ik}W_{kj} - W_{ik}S_{kj}) \\ +G_3(S_{ik}S_{kj} - \frac{1}{3}\eta_1\delta_{ij}), \end{aligned} \tag{15}$$

where,

$$\eta_1 = S_{ij}S_{ij}; \text{ and } \eta_2 = W_{ij}W_{ij}. \tag{16}$$

In the above equations $G_1 - G_3$ are yet to be determined scalar functions of the invariants of strain and rotation rate tensors. During the evolution of b_{ij}, $G_1 - G_3$ will be functions of time as well. Also note that, by definition, $\eta_1 + \eta_2 = 1$.

The representation for b_{ij} is now substituted into equation (12) and the resulting equation is simplified using the following identities valid for all two-dimensional mean flows:

$$\begin{aligned} S_{ik}S_{kj} &= \frac{1}{2}\eta_1\delta_{ij}^{(2)}; W_{ik}W_{kj} = -\frac{1}{2}\eta_2\delta_{ij}^{(2)}; \\ S_{ik}S_{kl}S_{lj} &= \frac{1}{2}\eta_1 S_{ij}; S_{ik}W_{kl}S_{lj} = -\frac{1}{2}\eta_1 W_{ij}; \\ W_{ik}S_{kl}W_{lj} &= \frac{1}{2}\eta_2 S_{ij}; S_{mn}b_{mn} = G_1\eta_1, \end{aligned} \tag{17}$$

where $\delta_{ij}^{(2)}$ and δ_{ij} are the two and three dimensional delta functions respectively. We retain terms up to quadratic power (in strain and rotation rate) in their original form and invoke their two-dimensional property only when these terms appear in cubic and higher power terms. For example, $S_{ik}S_{kj}$ is retained as such when it appears by itself: whereas, when it appears as a part of a cubic or higher power term we invoke $S_{ik}S_{kj} = \frac{1}{2}\eta_1\delta_{ij}^{(2)}$ to write $S_{ik}S_{kj}S_{jl} = \frac{1}{2}\eta_1 S_{il}$. By invoking the two-dimensional property for reducing only cubic and higher power terms, it is hoped that the three-dimensional effect is approximately accounted for upto the quadratic term. Using these rules, we write

$$\begin{aligned} S_{ik}b_{kj} + b_{ik}S_{kj} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij} &= \frac{1}{3}\eta_1 G_3 S_{ij} \\ &+2G_1(S_{ik}S_{kj} - \frac{1}{3}\eta_1\delta_{ij}) \\ W_{ik}b_{kj} - b_{ik}W_{kj} &= -G_1(S_{ik}W_{kj} - W_{ik}S_{kj}) \\ &+2\eta_2 G_2 S_{ij}. \end{aligned} \tag{18}$$

Substitution of equation (15) into equation (12) yields the anisotropy evolution equation in terms of coefficients G_1 - G_3. After substitution, the coefficient of each tensor on either side of equation (12) has to be equal due to the linear independence of the generators of the integrity basis. Comparing the coefficients of the three representation tensors on either side we get

$$\frac{dG_1}{dt} + G_1(L_1^0\omega - L_1^1 G_1\eta_1) = L_2 + \frac{1}{3}L_3 G_3\eta_1 \tag{19}$$

$$
\begin{aligned}
&+2L_4\eta_2 G_2 \\
\frac{dG_2}{dt} &+ G_2(L_1^0\omega - L_1^1 G_1\eta_1) = -G_1 L_4, \\
\frac{dG_3}{dt} &+ G_3(L_1^0\omega - L_1^1 G_1\eta_1) = 2G_1 L_3.
\end{aligned}
$$

Equations (19) along with (13) constitute the new non-linear system of evolution equations for Reynolds stresses in homogeneous turbulence.

3. Dynamical System Analysis. Fixed point and bifurcation analyses of the new system is now performed.

The fixed point equations (14) can now be restated as

$$
\frac{dG_1}{dt} = \frac{dG_2}{dt} = \frac{dG_3}{dt} = \frac{d\omega}{dt} = 0. \tag{20}
$$

Using the notation that the fixed point values are denoted by a superscript 0, the algebraic fixed point relations are (using $b_{mn}S_{mn} = G_1\eta_1$):

$$
\begin{aligned}
0 &= 2\omega^0(C_{e1}-1)G_1^0\eta_1 + (C_{e2-1}-1)(\omega^0)^2 \\
0 &= -G_1^0(L_1^0\omega - L_1^1 G_1^0\eta_1) + L_2 + \frac{1}{3}L_3 G_3^0\eta_1 \\
&\quad +2L_4\eta_2 G_2^0 \\
0 &= G_2^0(L_1^0\omega - L_1^1 G_1^0\eta_1) + G_1^0 L_4 \\
0 &= G_3^0(L_1^0\omega - L_1^1 G_1^0\eta_1) - 2G_1^0 L_3
\end{aligned} \tag{21}
$$

This system of equations leads to the five fixed points:

$[\omega^0 = 0,\ G_1^0 = 0,\ L_2 + \frac{1}{3}L_3 G_3\eta_1 + 2L_4\eta_2 G_2 = 0]$;

$[\omega^0 = 0,\ G_1^0 = -\frac{1}{\sqrt{\eta_1}}Q_1,\ G_2^0 = \frac{L_4}{L^*\eta_1},\ G_3^0 = -\frac{2L_3}{L^*\eta_1}]$;

$[\omega^0 = 0,\ G_1^0 = +\frac{1}{\sqrt{\eta_1}}Q_1,\ G_2^0 = \frac{L_4}{L^*\eta_1},\ G_3^0 = -\frac{2L_3}{L^*\eta_1}]$;

$[\omega^0 = -2\frac{C_{e1}-1}{C_{e2}-1}G_1^0\eta_1,\ G_1^0 = -\frac{Q^*}{\sqrt{\eta_1}},\ G_2^0 = \frac{L_4}{L^*\eta_1},\ G_3^0 = -\frac{2L_3}{L^*\eta_1}]$;

$[\omega^0 = -2\frac{C_{e1}-1}{C_{e2}-1}G_1^0\eta_1,\ G_1^0 = +\frac{Q^*}{\sqrt{\eta_1}},\ G_2^0 = \frac{L_4}{L^*\eta_1},\ G_3^0 = -\frac{2L_3}{L^*\eta_1}]$.

In the above equations Q_1 and Q^* are defined as

$$
\begin{aligned}
Q_1 &= \sqrt{-\frac{L_2}{L_1^1} + \frac{2}{3}(\frac{L_3}{L_1^1})^2 - 2(\frac{L_4}{L_1^1})^2\frac{1-\eta_1}{\eta_1}}, \\
Q^* &= \sqrt{-\frac{L_2}{L^*} + \frac{2}{3}\frac{L_3^2}{L^{*2}} - 2\frac{L_4^2}{L^{*2}}\frac{1-\eta_1}{\eta_1}},
\end{aligned} \tag{22}
$$

where

$$
L^* = 2L_1^0\frac{C_{e1}-1}{C_{e2}-1} + L_1^1. \tag{23}
$$

It should be pointed out that a negative value of G_1^0 is consistent with a gradient-diffusion type effect and energy flow from the mean to the fluctuating velocity field. A positive value would imply count-gradient diffusion and a negative value of production – i.e, flow of energy from the turbulent fluctuations to the mean flow.

3.1. Bifurcation analysis. The qualitative behavior of the solution of a set of differential equations may depend on the parameters of the system. For example, the nature and even the number of fixed points of a system can change with changing parameter values (bifurcation). The parameters of the present system of equations are the constants of the pressure-strain correlation model and the mean strain-rate to total deformation ratio η_1. (Note that, by definition, $\eta_2 = 1 - \eta_1$, and hence η_2 is not considered an independent parameter.) In this paper, we restrict our attention to bifurcation due to η_1 alone for a given pressure-strain correlation model.

The system has five fixed points when η_1 is such that both Q_1 and Q^* are real. If only one of Q_1 or Q^* is real, then the system has three fixed points. If both are not real, then there is only one fixed point. It is important to establish the conditions under which Q_1 and Q^* are real.

For Q_1 to be real we require

$$-\frac{L_2}{L_1^1} + \frac{2}{3}(\frac{L_3}{L_1^1})^2 - 2(\frac{L_4}{L_1^1})^2 \frac{1-\eta_1}{\eta_1} \geq 0, \tag{24}$$

implying that η_1 should be greater than a critical value η_1^a:

$$\eta_1 \geq \eta_1^a = \frac{2L_4^2}{-L_2 L_1^1 + \frac{2}{3}L_3^2 + 2L_4^2}. \tag{25}$$

Similarly, Q^* is real only when

$$\eta_1 \geq \eta_1^b = \frac{2L_4^2}{-L_2 L^* + \frac{2}{3}L_3^2 + 2L_4^2}. \tag{26}$$

For all the pressure-strain correlation models considered in this paper, L_2 is negative and $L^* > L_1^1$ leading to

$$1 > \eta_1^a > \eta_1^b > 0. \tag{27}$$

When η_1 is in the interval $(1, \eta_1^a)$, the system has five fixed points. The system has three fixed points in the interval (η_1^a, η_1^b). Finally, for $\eta_1 < \eta_1^b$, the system has only one fixed point.

In summary, the nonlinear set of equations governing the (modeled) evolution of the Reynolds stress has two bifurcation points η_1^a and η_1^b. The behavior of the solution for various values of η_1 depends upon the stability of the various fixed points.

3.2. Stability of fixed points. In order to establish the stability of a fixed point, we need to examine if any small perturbation of the system away from the fixed point eventually returns to the fixed point after a sufficiently long time. The most expeditious way of establishing this is by investigating the Jacobian of the system at the fixed point. If an eigenvalue of the Jacobian is negative, solution trajectories are attracted towards the fixed point along the corresponding eigenvector. On the other hand, if an eigenvalue is positive, the solution trajectory is repelled away from the fixed point along the corresponding eigenvector. For a fixed point to be stable all the eigenvalues must be negative, so that all trajectories in the neighborhood of the fixed point are attracted towards the fixed point. If all permissible initial conditions are attracted to the fixed point, such a fixed point is called globally asymptotically stable. The set of all initial conditions that ultimately evolve to a stable fixed point is called the basin of attraction of that fixed point. For a nonlinear set of equations, such as the one considered here, it is difficult to establish the basin of attraction and will not be attempted here. We will only seek to establish the local asymptotic stability of each of the fixed points. First, the various types of fixed points are listed.

1. When all the eigenvalues of the Jacobian are real and negative, then the fixed point is a stable fixed point also called an attractor or a sink.
2. If all the eigenvalues are real and positive, then the fixed point is a source or repellor. All solution trajectories in the neighborhood of the fixed point are repelled away from it.
3. If the eigenvalues are real with some positive and the others negative, then the fixed point is of the saddle type. The solution trajectories are attracted towards the fixed point in some directions and repelled away in other directions.
4. If the eigenvalues are complex and the real part is positive, then the fixed point is a spiral source. The solution to the system fluctuates about the fixed point with the amplitude of the fluctuation getting larger with time.
5. If the eigenvalues are complex with a negative real part, then the fixed point is a spiral sink. The solution is oscillatory with decreasing magnitude about the fixed point.
6. If the eigenvalues are purely imaginary, then the fixed point is classified as a center. The asymptotic solution then displays an oscillatory behavior.

The sink and spiral sink fixed points are attracting fixed points and, hence, stable. Saddle and source fixed points are unstable.

Fixed point # 1.. This fixed point is given by

$$\omega^0 = 0,\;\; G_1^0 = 0,\;\; L_2 + \frac{1}{3} L_3 G_3 \eta_1 + 2L_4 \eta_2 G_2 = 0.$$

This is the only fixed point that exists for the entire range of η_1 values.

The Jacobian at this fixed point is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2L_4(1-\eta_1) & \frac{1}{3}L_3\eta_1 \\ 0 & L_1^1\eta_1 G_2^0 - L_4 & 0 & 0 \\ 0 & L_1^1\eta_1 G_3^0 + 2L_3 & 0 & 0 \end{pmatrix} \tag{28}$$

The eigenvalues of the Jacobian are

$$\lambda_1 = 0,\;\; \lambda_2 = 0,\;\; \lambda_3 = +\sqrt{\eta_1} L_1^1 Q,\;\; \lambda_4 = -\sqrt{\eta_1} L_1^1 Q,$$

and the corresponding eigenvectors are

$$\begin{aligned} \mathbf{v}_1 &= [1,0,0,0], \\ \mathbf{v}_2 &= [0,0,-\frac{L_3\eta_1}{6L_4(1-\eta_1)},1], \\ \mathbf{v}_3 &= [0,+\sqrt{\eta_1}L_1^1 Q, L_1^1\eta_1 G_2^0 - L_4, L_1^1\eta_1 G_3^0 + 2L_3], \\ \mathbf{v}_4 &= [0,-\sqrt{\eta_1}L_1^1 Q, L_1^1\eta_1 G_2^0 - L_4, L_1^1\eta_1 G_3^0 + 2L_3]. \end{aligned} \tag{29}$$

Since two of the eigenvalues are zero, this fixed point is classified as a non-hyperbolic fixed point. So long as Q is real ($\eta_1 \geq \eta_1^b$), λ_3 is positive and λ_4 is negative leading to this fixed point being a saddle and hence unstable. When $\eta_1 < \eta_1^b$ the eigenvalues λ_3 and λ_4 are both purely imaginary; the fixed point is a center and the asymptotic solution is oscillatory. The stability of non-hyperbolic center fixed point cannot be gleaned from a linear approximation. The stability can be evaluated only from a center manifold theory which is not attempted here. It suffices here to say that for $\eta_1 < \eta_1^b$, the long-time behavior of the Reynolds stress in oscillatory.

Fixed points #2 and #3.. These fixed points are given by

$$\omega^0 = 0,\ \ G_1^0 = \pm\frac{1}{\sqrt{\eta_1}}Q_1,\ \ G_2^0 = \frac{L_4}{L^*\eta_1},\ \ G_3^0 = -\frac{2L_3}{L^*\eta_1},$$

and exist only in the range $\eta_1 \geq \eta_1^a$. The Jacobian at this fixed point is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -G_1^0 & 2G^* & 2L_4(1-\eta_1) & \frac{1}{3}L_3\eta_1 \\ -G_1^0 & 0 & G^{**} & 0 \\ -G_1^0 & 0 & 0 & G^{**} \end{pmatrix} \tag{30}$$

where $G^* = G_1^0\eta_1(L_1^1 + C_{e1} - 1)$ and $G^{**} = G_1^0\eta_1(L_1^1 + 2C_{e1} - 2)$.

The eigenvalues are given by

$$\begin{aligned} \lambda_1 &= -2(C_{e1}-1)G_1^0\eta_1,\ \ \lambda_2 = 2L_1^1G_1^0\eta_1, \\ \lambda_3 &= \lambda_4 = L_1^1G_1^0\eta_1. \end{aligned} \tag{31}$$

The eigenvectors are,

$$\begin{aligned} \mathbf{v}_1 &= [\frac{\eta_1}{L_1^0}(L_1^1 + 2C_{e1} - 2), \\ &\quad \frac{G_1^0\eta_1(L_1^1 + 2C_{e2} - 2) + 2L_4\eta_2 + \frac{1}{3}L_3\eta_1}{2G_1^0\eta_1(L_1^1 + C_{e1} - 1)}, 1, 1], \\ \mathbf{v}_2 &= [0, 1, 0, 0], \\ \mathbf{v}_3 &= [0, -\frac{2L_4(1-\eta_1)}{G_1^0L_1^1\eta_1}, 1, 0], \\ \mathbf{v}_4 &= [0, \frac{L_3}{3G_1^0L_1^1}, 0, 1]. \end{aligned} \tag{32}$$

Since all the eigenvalues are non-zero this an hyperbolic fixed point. In all the models considered, L_1^1 and $(C_{e1} - 1)$ are both positive and η_1 is positive by definition. As a result, irrespective of the sign of G_1^0, some of the eigenvalues will be positive and others negative and the fixed point is a saddle. These two fixed points are unstable when $\eta_1 > \eta_1^a$ and do not exist otherwise. They do not play an important role in the long-time behavior of the Reynolds stresses.

Fixed point #4.. This fixed point is given by

$$\begin{aligned} \omega^0 &= -2\frac{C_{e1}-1}{C_{e2}-1}G_1^0\eta_1,\ \ G_1^0 = -\frac{1}{\sqrt{\eta_1}}Q^* \\ G_2^0 &= \frac{L_4}{L^*\eta_1},\ \ G_3^0 = -\frac{2L_3}{L^*\eta_1}. \end{aligned} \tag{33}$$

This fixed point exists only for $\eta_1 \geq \eta_1^b$. Due to the complex nature of the Jacobian, it is difficult to obtain all the eigenvalues and eigenvectors symbolically. However, one eigenvalue is easily obtained by inspection:

$$\lambda_1 = G_1^0\eta_1L^*;\ \mathbf{v}_1 = [0, 0, 1, -6\frac{L_4(1-\eta_1)}{L_3\eta_1}]. \tag{34}$$

The eigenvalues evaluated numerically are plotted in Figure 1(a) as a function of the parameter η_1 for the linearized-SSG pressure-strain correlation model. All of the eigenvalues are non-zero (hyperbolic fixed point) and negative indicating that this is an attractor. (Note that the quantity plotted is the negative of the actual eigenvalues.) Another important point to be gleaned from the the figure is that eigenvalue λ_4 is always about an order of magnitude smaller than the other eigenvalues. This indicates that the evolution equations evolve slowly along the eigenvector associated with λ_4 and rapidly along all other direction. The eigenvector direction corresponding to λ_4 is shown in Figure 1(b) and it is almost coincident with the ω axis. The behavior of the eigenvalues of this fixed point with LRR pressure-strain correlation model is qualitatively and, even, quantitatively similar to that of SSG model.

Fixed point #5.. This fixed point

$$\omega^0 = -2\frac{C_{e1}-1}{C_{e2}-1}G_1^0\eta_1, \quad G_1^0 = +\frac{1}{\sqrt{\eta_1}}Q^* \tag{35}$$

$$G_2^0 = \frac{L_4}{L^*\eta_1}, \quad G_3^0 = -\frac{2L_3}{L^*\eta_1}.$$

also exists only for $\eta_1 \geq \eta_1^b$. The eigenvalues calculated numerically are all positive indicating that the fixed point is a source or a repellor and, hence, unstable.

All the stability and bifurcation results for G_1 are summarized in Figure 2. The bifurcation diagram of ω is given in Figure 3.

4. Summary and Discussion. The long-time behavior of the standard Reynolds stress closure equation depends upon the value of the parameter η_1, the fraction of deformation that is strain. The asymptotic behavior undergoes bifurcation at $\eta_1 = \eta_1^b$. For values of η_1 higher than η_b, the solution of the equation set asymptotes monotonically to an attractor. For smaller values, the long-time behavior is oscillatory.

Asymptotic behavior for $\eta_1 > \eta_1^b$.. The solution is attracted to fixed point 4. The production to dissipation ratio at this equilibrium state is

$$\frac{P}{\varepsilon}(equilibrium) = \frac{C_{e2}-1}{C_{e1}-1} \tag{36}$$

independent of η_1. In appropriately normalized time, the turbulent kinetic energy and dissipation grow exponentially at rates independent of η_1:

$$\frac{d\ln K}{d\tau} = \frac{P}{\varepsilon} - 1; \quad \frac{d\ln\varepsilon}{d\tau} = C_{e1}\frac{P}{\varepsilon} - C_{e2}, \tag{37}$$

where $\tau = \omega_0 t$.

Longtime behavior for $\eta_1 < \eta_1^b$.. The long-time behavior in this case is dictated by fixed point 1 which is non-hyperbolic (has zero eigenvalues). Linear analysis about the fixed point is inadequate to determine its stability. Numerical calculations indicate that (i) ω decays monotonically (exponentially?) to zero, (ii) G_1 oscillates about zero, (iii) G_2 and G_3 converge to non-zero values, (iv) with decreasing η_1 the asymptotic growth rates of kinetic energy and dissipation decrease. In fact, for small enough η_1 values kinetic energy and dissipation *decay* in time leading to relaminarization.

Implications to pressure-strain correlation modeling.. It is reasonable to demand that future models yield bifurcation diagrams consistent with experimentally (laboratory and numerical) observed behavior of the Navier-Stokes equation. At the very least, for the sake of qualitative consistency with true physics, the bifurcation points of the model equation set should coincide with that of Navier-Stokes statistics.

The DNS data of Blaisdell and Shariff (1996) appears to indicate that the asymptotic growth rate of kinetic energy is always positive and nearly independent of η_1:

$$\frac{d\ln K}{dt} \approx Const.(>0), \ for\ 1 \geq \eta_1 > 0, \tag{38}$$

a result that is consistent with linear stability analysis. The lack of qualitative (or, even, quantitative) change in the behavior of the kinetic energy growth rate appears to imply that there is no bifurcation, contrary to model predictions. This can be interpreted as the bifurcation taking place at $\eta_1 = 0$. If this is indeed true (further work is currently underway), then, in order for the pressure strain model to be consistent with physics we need:

$$\eta_1^b = \frac{2L_4^2}{-L_2L^* + \frac{2}{3}L_3^2 + 2L_4^2} = 0, \quad (39)$$
$$\text{leading to } \ L_4 = 0, \ \text{ implying } \ C_4 = 2.$$

This should be the behavior of C_4 in the limit of vanishing values of η_1. The currently used values of C_4 yield good agreement with data in homogeneous shear and strain dominated cases. This clearly suggests that C_4 and perhaps all other coefficients should be functions of η_1 and η_2. These ideas will be explored in future works.

Non-equilibrium algebraic stress modeling.. Non-equilibrium algebraic Reynolds stress model can be considered as the approximate solution of the Reynolds stress closure equations away from the equilibrium state. When $\eta_1 > \eta_1^b$, the solution approaches the equilibrium state (fixed point 4) along the invariant manifold corresponding to eigenvector $\mathbf{v}_4$ for nearly all initial conditions. This is because the corresponding eigenvalue (λ_4) is much smaller than the rest (Figure 1a). (From a random initial condition, the solution evolution along the other eigenvector directions is very rapid, so that the trajectory after a short initial stage is nearly aligned with the slowest eigen-direction.) In Girimaji (1996b), it was suggested that the invariant manifold of $\mathbf{v}_4$ be used as the non-equilibrium Reynolds stress model and a strategy for determining the manifold was presented. When $\eta_1 < \eta_1^b$, the strategy for non-equilibrium algebraic modeling in not yet clear. Central manifold reduction may be the answer and that approach is currently under investigation.

REFERENCES

R. Abid and C. G. Speziale (1993), *Predicting equilibrium states with Reynolds stress closure in channel and homogeneous shear flows.* Phys. Fluids A, 5, pp. 1776-1782.

G. A. Blaisdell and K. Shariff (1996), *Simulation and modeling of elliptic streamline flow.* Center for Turbulence Research, Proceedings of the Summer Program 1996.

M. M. Gibson and B. E. Launder (1978), *Ground effects of pressure fluctuations in the atmospheric boundary layer.* J. Fluid Mech., 86, pp 491–511.

S. S. Girimaji (1996a), *Fully explicit and self-consistent algebraic Reynolds stress model.* Theoret. Comput. Fluid Dynamics, Vol. 8, pp 387-402.

S. S. Girimaji (1996b), *Non-equilibrium algebraic Reynolds stress model.* Bulletin of the Amer. Physical Soc., 41 (9), pp 1813.

B. E. Launder, G. J. Reece and W. Rodi (1975), *Progress in the Development of Reynolds Stress Turbulence Closure.* J. Fluid Mech., 68, pp 537–566.

C. G. Speziale, R. Abid and G. A. Blaisdell (1996), *On the consistency of Reynolds stress turbulence closures with hydrodynamic stability theory.* Phys. Fluids, 8, pp 781–788.

C. G. Speziale and N. Mac Giolla Mhuiris (1989), *On the prediction of the equilibrium states in homogeneous turbulence*. J. Fluid Mech., 209, pp. 591-615.

C. G. Speziale and N. Mac Giolla Mhuiris (1990), *Scaling laws for homogeneous turbulent shear flows in a rotating frame*. Phys. Fluids A, 1, pp. 294-301.

C. G. Speziale, S. Sarkar and T. B. Gatski (1991), *Modeling the pressure-strain correlation of turbulence: an invariant dynamical system approach*. J. Fluid Mech., 227, pp. 245–272.

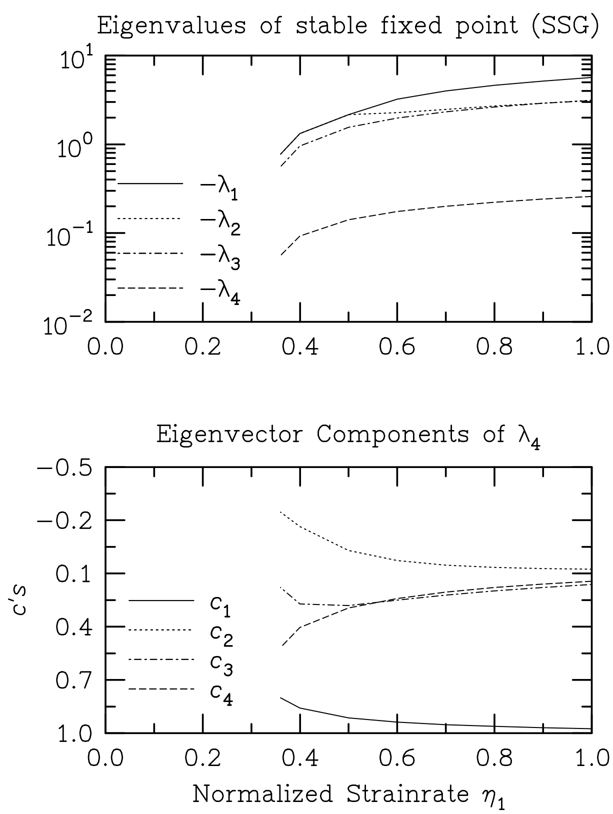

Fig. 1. (a) Eigenvalues of fixed point 4 as a function of η. (b) Components of eigenvector corresponding to eigenvalue λ_4.

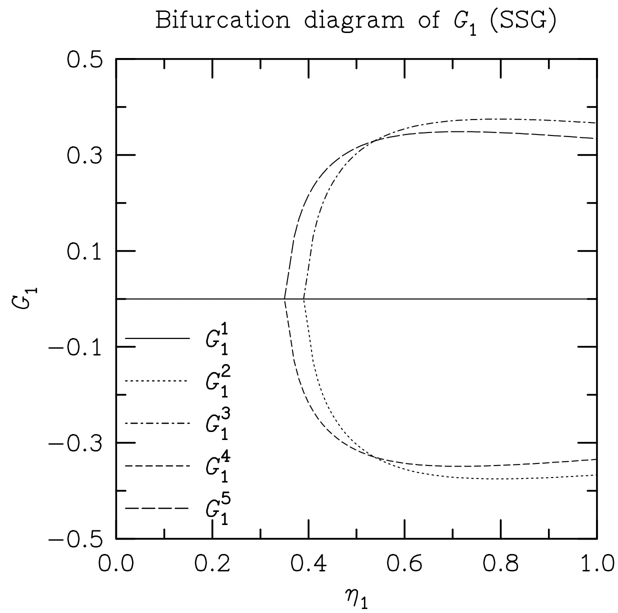

Fig. 2. Bifurcation diagram of G_1. Fixed point G_1^1: unstable (saddle) for $\eta \geq \eta_1^b$ and center for $\eta < \eta_1^b$. G_2, G_3: unstable (saddle). G_4: stable (attractor). G_5: unstable (repellor).

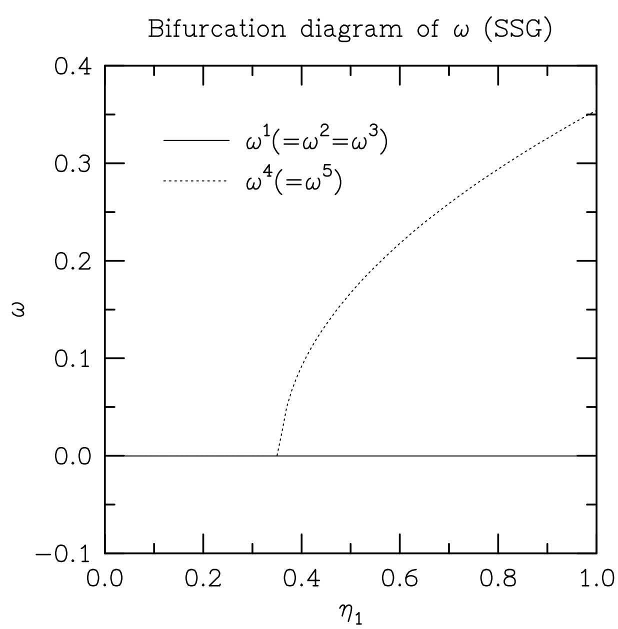

Fig. 3. Bifurcation diagram of ω. Fixed point $\omega^1(=\omega^2=\omega^3)$ is unstable (saddle) for $\eta \geq \eta 1_1^b$ and stable for $\eta < \eta_1^b$. $\omega^4(=\omega^5)$ is stable (attractor).