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ON THE BEHAVIOR OF VELOCITY FLUCTUATIONS IN RAPIDLY ROTATING FLOWS

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Abstract. The behavior of velocity fluctuations subjected to rapid rotation is examined. The rapid rotation considered is any arbitrary combination of two basic forms of rotation, reference frame rotation and mean flow rotation. It is recognized that the two types of rotating flows differ in the manner in which the fluctuating fields are advected. The first category is comprised of flows in rotating systems of which synoptic scale geophysical flows are a good example. In this class of flows the fluctuating velocity field advects and rotates *with* the mean flow. In the rapid rotation limit, the Taylor-Proudman theorem describes the behavior of this class of fluctuations. Velocity fluctuations that are advected *without* rotation by the mean flow constitute the second category which includes vortical flows of aerodynamic interest. The Taylor-Proudman theorem is not pertinent to this class flows and a new result appropriate to this second category of fluctuations is derived. demonstrates that general fluctuating velocity fields are rendered two-dimensional and horizontally non-divergent in the limit of any large combination of reference frame rotation and mean-flow rotation. The concommitant 'geostrophic' balance of the momentum equation is, however, dependent upon the form of rapid rotation. It is also demonstrated that the evolution equations of a two-dimensional fluctuating velocity fields are frame-indifferent with any imposed mean-flow rotation. The analyses and results of this paper highlight many fundamental aspects of rotating flows and have important consequences for their turbulence closures in inertial and non-inertial frames.

Keywords: Taylor-Proudman theorem, rotating turbulence, vortical flows

Subject classification: Fluid Mechanics

1. Introduction. Many important flows of geophysical and technological interest are strongly influenced by rotation. It is generally believed that the fluctuating velocity fields in rapidly rotating flows are described by the Taylor-Proudman theorem (Greenspan, 1968, Tritton, 1977). The Taylor-Proudman theorem and one of its consequences, the geostrophic balance, have long been used successfully in many geophysical applications (Pedlosky, 1982).

Flow rotation can take many forms most basic of which are reference frame rotation and mean flow rotation (vorticity). In the former case, the flow occurs in a rotating reference frame, whereas, the latter category corresponds to an imposed mean flow rotation. A general rotating flow can be an arbitrary combination of these two basic flow types. In this article, the term rotating flow is used describe any flow that

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is influenced by rotation; the terms reference frame rotation and mean flow rotation are the terms used to indicate the two basic types of rotation. The distinction between the two basic forms of rotating flow is best understood by considering one example of each flow: the homogeneous rotating shear flow and the elliptic streamline flow (Blaisdell and Sharif, 1996). These are two benchmark flows used to study the effect of rotation on turbulent fluctuations using direct numerical simulations (DNS). In the rotating homogeneous shear, the velocity fluctuations are subjected to a constant mean shear in a rotating reference frame. An elliptic flow is a homogeneous flow subjected to a constant mean strain and rotation in an inertial frame. The elliptic flow derives its name from the elliptic shape of the mean streamlines when the magnitude of mean rotation exceeds that of the mean strain. In the absence of strain, both the rotating and elliptic flows have circular streamlines. When the magnitude of strain and rotation are equal, the flow is a pure shear and the streamlines are rectilinear.

Despite the apparent similarities between the flows with reference frame rotation and flows with mean flow rotation, the fluctuating velocity fields behave very differently, Blaisdell and Sharif (1996). The difference between the two flows increases as the rotation rate increases despite the fact that the mean streamlines in both cases become circular. An important kinematic distinction between the fluctuating fields in the two cases can be made in an inertial frame. In the rotating homogeneous shear *the principal axes of the Reynolds stresses rotate at the rate of the mean rotation.* In the elliptic flow, *the principal axes of the Reynolds stresses are fixed.* It is an explanation of this physical observation that motivates aspects of this article.

The present state of development of the mechanics of rotating fluids does not distinguish between these two types of rotating flows. One important practical consequence of this fact is that the turbulence models calibrated in one class performs poorly in the other, Blaisdell and Sharif (1996). There is clearly a strong motivation for understanding the fundamental difference between the various classes of rotating flows and enhancing our ability to model the effect of rotation.

The objective of this paper is to examine the behavior of various types of rotating flows in the limit of rapid rotation. The article is concerned with fundamentals; the mathematical distinction and physical interpretation of the various flow types. It is first recognized that the fundamental difference between the various flows lies, as is suggested by the physical observation alluded to above, in the manner in which the fluctuating fields are advected. Three types of fluctuations are identified. In the limit of rapid mean flow rotation, we establish for each type (i) the mathematical description and (ii) its distinctive kinematical and dynamical properties. The new results highlight some important fundamental aspects of rotating flows and have important mathematical and phenomenological implications for their turbulence modeling.

In Section 2 the fluctuating energy equation is examined in the limit of rapid rotation to derive suitable approximations for the advection of fluctuating velocity and vorticity. In Section 3, the implications of these approximations on the fluctuation vorticity and momentum equation are derived for each flow class. The effect of reference-frame rotation and mean-flow rotation on two-dimensional fluctuations is examined in Section 4. Important turbulence modeling implications and physical insights are discussed in Section 5. We conclude with a summary of the results in Section 6.

2. Rapid rotation: velocity fluctuation magnitude. We first demonstrate that the magnitude of the velocity fluctuations are, in the limit of rapid rotation, advected without change. This asymptotic result is then used to derive the permissible representations for the advection of the velocity and vorticity fields in the rapid rotation limit. Establishing the form of the advection of the velocity immediately suggests a typology for rotating flows based on the rotation of the fluctuating velocity field or, as will be shown, on the rotation of the principal axes of the Reynolds stresses. This is a more unique characterization than the current distinctions of flow types based on the mean flow rotation or frame rotation. A classification of rotating flows into reference frame rotation and mean flow rotation is undesirable in as much as it involves the definition of the reference frame in the distinction. Moreover, such a classification cannot distinguish between the two flows when both are considered in an inertial reference frame.

The evolution equation for the fluctuating velocity implied by the Navier-Stokes equations in a non-inertial reference frame rotating at a rate Ω_i, is given by:

$$\frac{D}{Dt}u_i + u_j u_{i,j} + (W_{ij} + 2\varepsilon_{ikj}\Omega_k)u_j + \; S_{ij}u_j = \; -p_{,i} + \nu \; u_{i,jj} \; , \tag{1}$$

where $\frac{D}{Dt}$ represents the advection following a mean fluid particle:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_j \frac{\partial u_i}{\partial x_j}. \tag{2}$$

Upper case letters represent mean quantities and lower case indicates fluctuations. The pressure in normalized by the constant density. The kinematic viscosity is given by ν and ε_{ijk} represents the alternating tensor. Subscript $_{,j}$, represents partial derivative with respect to the j direction. Repeated indices imply summation. The mean strain rate, S_{ij}, and the mean rotation rate, W_{ij}, are given by

$$S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}); \quad W_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i}). \tag{3}$$

The mean intrinsic vorticity $\mathcal{W}_{ij}$ is defined as the sum of the system rotation and mean flow rotation:

$$\mathcal{W}_{ij} = W_{ij} + 2\varepsilon_{ikj}\Omega_k. \tag{4}$$

The equation for the magnitude of the velocity fluctuations is obtained from equation (1):

$$\begin{aligned} \frac{1}{2}[\frac{D}{Dt}(u_i u_i) + u_j(u_i u_i)_{,j}\,] &= -\mathcal{W}_{ij}u_j u_i - S_{ij}u_j u_i \; - \; p_{,i}\, u_i + \nu \, u_i u_{i,jj} \; . \\ &= -S_{ij}u_j u_i - p_{,i}\, u_i + \nu \, u_i u_{i,jj} \; . \end{aligned} \tag{5}$$

This equation will be used to show that in the limit of rapid rotation that the magnitude of the velocity fluctuations are approximately constant.

2.1. The rapid rotation limit. We are interested in the evolution rate of the magnitude of the velocity fluctuations in the limit of rapid rotation or, more precisely, large intrinsic vorticity. Throughout the paper, as in the case of the Taylor-Proudman theorem, the rapid rotation limit implies:

$$\frac{S}{\mathcal{W}} << 1; \qquad \frac{\omega}{\mathcal{W}} << 1, \tag{6}$$

where, S, $\mathcal{W}$ and ω represent, respectively, the norms of the mean strain rate, intrinsic mean vorticity and fluctuating vorticity. The square of the fluctuating vorticity, ω^2 can also be interpreted as the energy cascade rate. The large intrinsic vorticity analysis includes, of course, that rapid reference frame rotation as well as the large mean flow rotation cases. Our analyses are to be understood, in the same sense as the Taylor-Proudman theorem, as an asymptotic analysis in the limit specified.

On normalizing by the intrinsic mean vorticity, the equation for the magnitude of the velocity fluctuations becomes

$$\frac{1}{2\mathcal{W}}[\frac{D}{Dt}(u_i u_i) + u_j(u_i u_i)_{,j}\,] = -\,\frac{1}{\mathcal{W}}[S_{ij}u_j u_i \ - \ (pu_i)_{,i} + \ \nu \ u_i u_{i,jj}\,]. \tag{7}$$

The terms on the right hand side of the above correspond to the production, pressure transport, and viscous dissipation of the energy of the fluctuations. The production and dissipation will scale with the cascade rate, which in the rapid rotation limit, is small (equation 6). The turbulent transport terms do not contribute towards the overall growth of fluctuating energy as they merely transport energy from locations of production to areas of dissipation. If production and dissipation are individually small, these effects must be small. Thus, on the time scale of the intrinsic vorticity

$$\frac{1}{\mathcal{W}}u_j u_{i,j} \rightarrow \ 0; \quad \frac{1}{\mathcal{W}}S_{ij}u_i u_j \rightarrow \ 0; \quad \frac{1}{\mathcal{W}}\nu u_i \ u_{i,jj} \rightarrow \ 0; \quad \frac{1}{\mathcal{W}}(pu_i)_{,i} \rightarrow \ 0. \tag{8}$$

Thus, on the time scale of the intrinsic vorticity the magnitude of fluctuating velocity following a fluid particle changes slowly:

$$\frac{1}{\mathcal{W}}\frac{D}{Dt}(u_i u_i) \rightarrow \ 0. \tag{9}$$

That is, the magnitude of the velocity is approximately constant. Given that the magnitude of the velocity is approximately constant, the advection must be of the general form

$$\frac{D}{Dt}u_i \approx \mathcal{R}_{ij}u_j, \tag{10}$$

where $\mathcal{R}_{ij}$ is a general antisymmetric tensor so that, in forming the energy equation, $\mathcal{R}_{ij}u_j u_i = 0$. The form of (10) is, of course, recognized as the pure rotation of a vector field.

The arbitrary rotation tensor $\mathcal{R}_{ij}$ can be decomposed into mean and fluctuating :

$$\mathcal{R}_{ij} = R_{ij} + r_{ij}, \qquad \langle r_{ij} \rangle = 0, \tag{11}$$

where R_{ij} represents a systematic (mean) rotation of the velocity field while r_{ij} represents random rotations about the mean rotation by the fluctuating field. The mean rotation rate can be expected to scale with the mean intrinsic vorticity and the random rotation with the fluctuating vorticity:

$$\frac{|r|}{|R|} \sim \frac{|\omega|}{\mathcal{W}} \ll 1, \tag{12}$$

as is consistent with the inequalities already assumed. Therefore, on the timescale of intrinsic vorticity,

$$\frac{D}{Dt}u_i \approx R_{ij}u_j. \tag{13}$$

This equation, describing the pure rotation of the velocity field at some as of yet unspecified rotation rate, is the fundamental equation from which we shall obtain several results pertinent to flows undergoing rapid rotation. The rotation rate R_{ij} will be called the kinematic rotation tensor.

The implication of equation (13) for the rotation of Reynolds stresses is straightforward:

$$\frac{D}{Dt}\langle u_i u_j \rangle = R_{ik}\langle u_j u_k \rangle + R_{jk}\langle u_i u_k \rangle. \tag{14}$$

The kinematic rotation rate of the fluctuating velocity field is, on the time scale of the rotation, equal to the rotation rate of the principal axes of the Reynolds stresses. Thus, statistically, R_{ij} can be understood or even called the rotation rate of the Reynolds stresses.

The implication of equation (13) for the fluctuating vorticity is easily obtained by taking its curl:

$$\frac{D}{Dt}\omega_i \approx \frac{1}{2}(W_j - R_j)u_{i,j} + W_{ij}\omega_j = \frac{1}{2}(W_j u_{j,i} - R_j u_{i,j}). \tag{15}$$

Note that the advection of the vorticity is *not* a simple rotation: there are very complicated production terms involving the fluctuating velocity gradients. In the above equation, the following relationships between an antisymmetric tensor and its associated axial vector have been used:

$$\begin{aligned} W_i &= \varepsilon_{ijk}W_{kj}; \quad W_{ij} = -\frac{1}{2}\varepsilon_{ijk}W_k; \\ \omega_i &= \varepsilon_{ijk}\omega_{kj}; \quad \omega_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k; \\ R_i &= \varepsilon_{ijk}R_{kj}; \quad R_{ij} = -\frac{1}{2}\varepsilon_{ijk}R_k. \end{aligned} \tag{16}$$

To summarize: In the limit of rapid rotation we have shown that the advection of the velocity approaches a pure rotation. There are two important implications of this result. The first is that the rotation of the principal axes of the Reynolds stresses is a very close approximation to the rotation rate of the velocity field. Secondly, an expression for the advection of the vorticity, consistent with a pure rotation of the velocity field, can be obtained. The advection of vorticity is not a simple rotation: there are source terms that may contribute to the growth of enstrophy. Equations (13) and (15) form the foundation of this paper, as they will be used in the next section in the momentum and vorticity equations to infer important properties of the fluctuating fields in the limit of rapid rotation.

2.2. Types of kinematic rotations. In the limit of rapid rotation it was shown that the advection of the velocity field is given by a pure rotation,

$$\frac{D}{Dt}u_i \approx R_{ij}u_j \tag{17}$$

where R_{ij}, by definition, is the rotation rate of the Reynolds stress tensor. We will now consider the values of R_{ij} associated with reference frame rotation, mean-flow rotation and arbitrary rotation. The value of R_{ij} will depend upon the reference frame of observation. In the inertial frame, it will be seen that $R_{ij} = 0$ corresponds to mean-flow rotation and $R_{ij} = W_{ij}$ corresponds to reference frame rotation. Knowing the kinematic rotation rate tensor, we obtain expressions for the advective derivatives of velocity and vorticity for the various flow classes. These expressions will be used to approximate the advective derivatives in the rapid rotation analysis of the vorticity and momentum equations.

Reference-frame rotation. It is known from observations of rotating shear flows that the Reynolds stress axes of this type of flow are fixed in a rotating reference frame. Therefore,

$$R_{ij} = 0 \qquad \Rightarrow \qquad [\frac{D}{Dt}u_i]_R \approx 0, \quad [\frac{D}{Dt}\omega_i]_R \approx 0. \tag{18}$$

In an inertial frame, the Reynolds stress tensor rotates with the mean flow at the rate of mean vorticity. If W_{ij} is the mean flow rotation tensor in an inertial frame then

$$R_{ij} = W_{ij} \quad \Rightarrow \quad \frac{D}{Dt}u_i \approx W_{ij}u_j, \quad \frac{D}{Dt}\omega_i \approx W_{ij}\omega_j = \frac{1}{2}W_j\omega_{j,i}. \tag{19}$$

As these fluctuations are Advected With the Mean flow Rotation they will be denoted AWMR fluctuations.

The mean Oldroyd derivative is a form of the substantial derivative convecting *and* rotating with the mean flow. The advective derivative in a rotating reference frame is exactly equal to the Oldroyd derivative in an inertial frame. For the AWMR fluctuations, the advective derivative is negligible in the rotating frame and the Oldroyd derivative is negligible in the inertial frame.

Mean-flow rotation. In the elliptic mean flows already mentioned, it has been observed that the principal directions of the Reynolds stress are fixed in an inertial frame. The rotation rate of the Reynolds stress is zero, thus

$$R_{ij} = 0 \quad \Rightarrow \quad \frac{D}{Dt}u_i \approx 0, \quad \frac{D}{Dt}\omega_i \approx \frac{1}{2}W_j u_{i,j} + W_{ij}\omega_j = \frac{1}{2}W_j u_{j,i}. \tag{20}$$

As these fluctuations are Advected With Out the Mean flow Rotation they will be denoted AWOR fluctuations.

General mean flows. Fluctuations in the presence of arbitrary combinations of reference frame rotation and mean flow rotation are of substantial interest. The rate of rotation of the Reynolds stresses is not, *a priori* known. Let the arbitrary rotation rate of the Reynolds stresses in a given frame of reference be denoted by R^a_{ij}. In that frame of reference, the fluctuating field description is

$$\begin{aligned} R_{ij} &= R^a_{ij}; \\ \frac{D}{Dt}u_i &\approx R^a_{ij}u_j; \\ \frac{D}{Dt}\omega_i &\approx \frac{1}{2}(W_j - R^a_j)u_{i,j} + W_{ij}\omega_j = \frac{1}{2}(W_j u_{j,i} - R^a_j u_{i,j}). \end{aligned} \tag{21}$$

This class of fluctuations will be called AWAR fluctuations since it represents Advection With Arbitrary Rotation.

Examples of different classes. The specification of a mean flow rotation does not constitute sufficient information to uniquely determine, *a priori*, the type of fluctuating velocity field. However, a classification of flow type is possible in many important flows based on experimental/numerical observation. Synoptic scale geophysical flows in the atmosphere are very visible examples of AWMR fluctuations. All flows in rotating systems, including many technological processes – crystal growth, centrifugal separation, rotating turbomachinery — are likely to fall into the AWMR category. In contrast, flows in stationary systems subject to mean flow rotation may belong to the AWOR category. Examples of such flows are many of the vortical aerodynamic flows — the trailing vortex, the flap-edge vortex, leading-edge vortex — and elliptic flows (Blaisdell and Shariff, 1996). Examples of AWAR category include elliptic flows in rotating reference frames, cylindrical Couette flow with the two cylinders rotating at different speeds and a cyclone which rotates with its characteristic angular speed on the rotating frame of the earth. In these flows, both reference-frame and mean-flow rotation are important.

3. Rapid rotation limit: vorticity and momentum equations. In the previous section a possible typology for rotating flows according to the value of the rotation tensor, R_{ij}, in the *ansatz* for the advective derivative of the velocity field, was indicated. This naturally suggested three different types of fluctuations which we have labelled AWMR, AWOR and AWAR. In this section the properties of AWMR, AWOR and AWAR fluctuations are, without regard to their physical origin, investigated. The kinematic *ansatz* for the advective derivative will now be used, for the first time, in the dynamical equations. It should be emphasized that the he objective of this section is not an *a priori* categorization of a particular rotating flow, but rather an *a posteriori* examination of important properties of the diverse fluctuating fields. The implications for the dynamical equations of diverse R_{ij} are explored.

3.1. AWMR fluctuations. In this subsection AWMR fluctuations are investigated. The analysis for the AWMR fluctuations is, in fact, a derivation of the well known Taylor-Proudman theorem, Proudman (1916), Taylor (1917), Greenspan (1968), Tritton (1977). This will serve as a familiar introduction and set precedent for procedures that will be used to obtain results in subsequent developments. The Taylor-Proudman result will be derived in the familiar rotating frame here and in an inertial frame in Section 3.3.

The incompressible Navier-Stokes equations in a non-inertial reference frame rotating with an angular velocity Ω_i^* is given by

$$\dot{u}_i^* + u_j^* u_i^*{}_{,j} + 2\varepsilon_{ikp}\Omega_k^* u_p^* = -p^*{}_{,i} + \nu u_i^*{}_{,jj}\,. \tag{22}$$

The instantaneous values of velocity and pressure are represented by u_i^* and p^* respectively. In this section *only*, asteriks will be used to represent dimensional quantities. Dimensionless variables are represented by the symbols without asteriks.

The equations are made nondimensional with length and velocity scales ℓ and u_c respectively. The time scale τ_c is formed from ℓ and u_c. The magnitude of the reference frame rotation rate is given by Ω_c. The Navier-Stokes equations become:

$$[\frac{1}{\tau_c\Omega_c}]\,\dot{u}_i + [Ro]\,u_j u_{i,j} + 2\varepsilon_{ikp}\Omega_k u_p = -\frac{1}{u_c\Omega_c}p_{,i} + [Ek]u_{i,jj}\,. \tag{23}$$

Here the Ekman number, $Ek = \frac{\nu}{\Omega_c\ell^2}$, and the Rossby number, $Ro = \frac{u_c}{\Omega_c\ell}$, represent, respectively, the relative importance of viscous and inertial forces to Coriolis forces. It is also useful to view the Rossby number as an indication of the magnitude of flow vorticity relative to the background vorticity due to frame rotation.

Rapid rotation limit of the vorticity equations.. Taking the curl of the Navier-Stokes equations one obtains the vorticity equation

$$[\frac{1}{\tau_c\Omega_c}]\,\dot{\omega}_i + [Ro]\,u_j\omega_{i,j} = [Ro]\,\omega_j u_{i,j} + 2\Omega_j u_{i,j} + [Ek]\,\omega_{i,jj}\,, \tag{24}$$

where $\omega_k = \varepsilon_{kiq}u_{i,q}$. In the limit of rapid rotation ($\Omega_c \to \infty$) the Rossby and Ekman numbers vanish and the fluctuating vorticity equation implied by the Navier-Stokes equation becomes

$$[\frac{1}{\tau_c\Omega_c}]\,\frac{D}{Dt}\omega_i \approx 2\Omega_j u_{i,j}\,. \tag{25}$$

The advection of the fluctuating vorticity for this class of fluctuations can be approximated to zero as given in equation (18). This approximation is valid for vorticity fluctuations whose *timescale* τ_c *is much larger than the timescale of the rotation of reference frame* $(1/\Omega_c)$, *leading to* $\tau_c\Omega_c \to \infty$ and

$$2\Omega_j u_{i,j} \to 0. \tag{26}$$

This is the Taylor-Proudman theorem and it is a statement valid only for those vorticity and velocity fluctuations that evolve slowly with respect to the frame rotation rate. It is valuable to consider the geophysical point of view. A flow structure consistent with $2\Omega_j u_{i,j} \rightarrow 0$ is necessary to keep, as is observed in nature, the relative flow vorticity (Rossby number) small.

The Taylor-Proudman result has several important consequences (Lighthill, 1966, Greenspan, 1968, Tritton, 1977). Without loss of generality assume that the axis of the frame rotation coincides with the '3' coordinate axis. The Taylor-Proudman result indicates that

$$\Omega_j u_{i,j} \rightarrow 0 \qquad \Rightarrow \qquad u_{i,3} = 0: \tag{27}$$

the fluctuating field is invariant along the axis of rotation. The fluctuating velocity field is two-dimensional $u_i = u_i(x_1, x_2)$. Since $u_{3,3} = 0$, the continuity equation yields

$$u_{1,1} + u_{2,2} = 0. \tag{28}$$

The flow is nondivergent in planes perpendicular to Ω_j.

Rapid rotation limit of the momentum equations. In the rapid frame rotation limit, the momentum equation reduces to

$$[\frac{1}{\tau_c \Omega_c}]\dot{u}_i + 2\varepsilon_{ikp}\Omega_k u_p = -\frac{1}{u_c \Omega_c}\, p_{,i}\,. \tag{29}$$

The advection of the velocity can be approximated as zero; for AWMR fluctuations $R_{ij} = 0$ in the rotating frame equation (18). More precisely, for fluctuations that evolve slowly in the rotating frame, $\tau_c \Omega_c \rightarrow \infty$ and $[\frac{1}{\tau_c \Omega_c}]\, \dot{u}_i \rightarrow 0$, leading to

$$2\varepsilon_{ikp}\Omega_k u_p = -\frac{1}{u_c \Omega_c}\, p_{,i}\,. \tag{30}$$

The scales of the fluctuating motion that evolve slowly in the rotating frame satisfy, to leading order, the geostrophic balance — a balance between Coriolis and pressure forces. Using the Poisson equation for pressure the pressure can be shown to scale as $p^* \sim u_c \Omega_c \ell$. As a consequence of the geostrophic balance the flow is described by a streamfunction. Multiplying the geostrophic balance by $\varepsilon_{iql}\Omega_l$ and contracting leads, in dimensional variables, to

$$u_p^* = \frac{1}{2}\varepsilon_{iqk}\, \frac{\Omega_k^*}{\Omega^2}\, p^*{}_{,i} + u_k^* \frac{\Omega_k^* \Omega_p^*}{\Omega^2}. \tag{31}$$

The fluctuating AMWR velocity field is, to leading order, described by a streamfunction:

$$u_p^* = \varepsilon_{pqk}\hat{\Omega}_k \psi_{,q} + w\delta_{i3}; \qquad \psi_{,p} = \varepsilon_{qpk}\hat{\Omega}_k u_q + \hat{\Omega}_p \hat{\Omega}_q \psi_{,q} \tag{32}$$

where $\psi = \psi\,(x, y)$. The streamfunction and pressure field are proportional; isobars and streamlines are parallel. Additional details can be found in Ristorcelli (1997).

In the present rotating coordinate system, AWOR fluctuations undergo a kinematic rotation. Therefore, AWOR fluctuations evolve on a time scale of the order of $\frac{1}{\Omega_c}$, leading to $\tau_c \Omega_c \rightarrow 1$. This is not consistent with $\tau_c \Omega_c \rightarrow \infty$ as required for the Taylor-Proudman theorem.

3.2. AWOR fluctuations. The AWOR fluctuations are now investigated. This is the type of fluctuations in which the velocity is advected but *not* rotated by the mean flow; $R_{ij} = 0$. These fluctuations are most conveniently analyzed in an inertial reference frame. The momentum equation for the fluctuations is given by

$$\frac{Du_i^*}{Dt} + \; u_j^* u_i^*{,_j} + u_k^* \, [S_{ik}^* + W_{ik}^*] = \; - p^*{,_i} \; + \nu \; u_i^*{,_{jj}} \; , \tag{33}$$

where $\frac{D}{Dt}$ indicates derivative following a mean fluid particle. The similarity with the momentum equations in the rotating frame (22) and (33) is clear: the mean vorticity now plays the role of the Coriolis force. It is important to note that the mean vorticity is twice the angular rotation: $W_k^* = 2\Omega_k$.

Rapid rotation limit of the vorticity equations. Taking the curl of the momentum equation produces the equation for fluctuating vorticity:

$$\frac{D\omega_i^*}{Dt} + u_j^* \omega_i^*{,_j} = \; \omega_j^* u_i^*{,_j} + W_j^* u_i^*{,_j} + \omega_j^* (S_{ij}^* + W_{ij}^*) + \; \nu \, \omega_i^*{,_{jj}} \; , \tag{34}$$

Normalizing the fluctuating vorticity equation with characteristic length, time, velocity scales, and characteristic mean rotation rate ℓ, τ_c, u_c, W_c produces

$$\begin{aligned} [\frac{1}{\tau_c W_c}] \frac{D\omega_i}{Dt} + [Ro_w] u_k \omega_i{,_k} = W_j u_i{,_j} + W_{ij} \omega_j + [\frac{S_c}{W_c}] S_{ij} \omega_j + \\ [Ro_w] \omega_j u_i{,_j} + [Ek_w] \, \omega_i{,_{jj}} \, . \end{aligned} \tag{35}$$

The corresponding Ekman and Rossby numbers are $Ek = \frac{\nu}{W_c \ell^2}$, and $Ro = \frac{u_c}{W_c \ell}$. Here $W_c^2 = W_{ij} W_{ij}$ and $S_c^2 = S_{ij} S_{ij}$. The ratio $[S_c/W_c]$ determines the ellipticity of the flow. In the limit rapid mean rotation or large mean vorticity, $W_c \to \infty$, the Ekman, Rossby and ellipticity parameters vanish and

$$[\frac{1}{\tau_c W_c}] \frac{D\omega_i}{Dt} \approx W_j u_i{,_j} + W_{ij} \omega_j \tag{36}$$

If the AWOR *ansatz*, equation (20), for the advective derivative of the vorticity fluctuations is applied, $\frac{D}{Dt}\omega_i = \frac{1}{2} W_j u_{j,i}$ one obtains the result

$$W_j u_i{,_j} \to 0. \tag{37}$$

This is equivalent to the Taylor-Proudman result with W_j replacing Ω_j. The velocity field is again two-dimensional being independent of the coordinate along the axis of the mean vorticity. The flow is also nondivergent in planes perpendicular to the rotation.

Rapid rotation limit of the momentum equations. While both AWMR and AWOR fluctuations are both independent of the coordinate along the axis of rotation and are both horizontally nondivergent, they are very different in other important aspects. This difference is seen in the form of the advective derivative which manifests itself dynamically in a different "geostrophic" balance of the momentum equations.

The momentum equations for the AWOR fluctuations, in the large mean vorticity limit, are

$$[\frac{1}{\tau_c W_c}] \, \frac{D}{Dt} u_i + u_k \, W_{ik} = - \, p_{,i} \, . \tag{38}$$

If the AWOR *ansatz*, equation (20), for the advective derivative of the velocity fluctuations is made one obtains the following balance

$$u_k \, W_{ik} = \frac{1}{2} \varepsilon_{ikq} W_q u_k \approx - \, p_{,i} \, . \tag{39}$$

Contracting the above balance with $\varepsilon_{ijq} W_q$ yields the streamfunction for the AWOR fluctuations:

$$u_j = 2\,\varepsilon_{ijq} W_q\, p_{,i} \; + u_q W_q W_j. \tag{40}$$

which in dimensional units becomes

$$u^*_q = \; 2\varepsilon_{ikq} \, \frac{W^*_k}{W^2_c} \, p^*{}_{,i} + u^*_k \frac{W^*_k W^*_q}{W^2_c}. \tag{41}$$

Pressure plays the role of a streamfunction as it did in the geophysical case. However, there is a factor of four difference in the prefactor of pressure in the two cases, equations (32) and (41). This reflects important dynamical differences between AWMR and AWOR fluctuations.

Consider the AWMR fluctuations of §3.1 in an inertial frame. Due to their rotation, they change rapidly on a timescale, τ_c, which is small $\tau_c \sim \frac{1}{W_c}$. For AWMR fluctuations, $\tau_c W_c \to 1$ and the analysis of the present section is not possible as $\tau_c W_c \to \infty$ is required. Thus AWMR and AWOR fluctuations, despite the same implications for the vorticity equation and a "twodimensionalization" of the flow are *distinct and mutually exclusive classes of velocity fields*.

3.3. AWAR fluctuations. The behavior of AWAR fluctuations in an inertial reference frame is now considered. The fluctuating velocity field evolves according to

$$\frac{D}{Dt} u^*_i + u^*_j u^*_i{}_{,j} + W^*_{ij} u^*_j + \; S^*_{ij} u^*_j = \; - p^*{}_{,i} + \nu \; u^*_i{}_{,jj} \, . \tag{42}$$

In the limit of rapid rotation, the effects of mean strain rate, viscosity and nonlinearity can be neglected leading to

$$\frac{D}{Dt} u^*_i + \; W^*_{ij} u^*_j \; \approx \; - p^*{}_{,i} \, . \tag{43}$$

The fluctuating vorticity equation in the rapid mean rotation limit is

$$\frac{D}{Dt} \omega^*_i = W^*_j u^*_i{}_{,j} + \omega^*_j W^*_{ij}. \tag{44}$$

These equations are now examined by invoking the approximations for the advective derivatives given in equation (21).

Rapid rotation limit of the vorticity equations. The advection of the AWAR vorticity fluctuations is approximated as, equation (21):

$$\frac{D}{Dt} \omega^*_i \approx \frac{1}{2} (W^*_j - R^a_j) u^*_{i,j} + W^*_{ij} \omega_j. \tag{45}$$

When this simplification is substituted into equation (44) we obtain:

$$(W^*_j + R^a_j) u^*_i{}_{,j} \approx 0, \tag{46}$$

for all combinations of W^*_j and R^a_j. If W^*_j and R^a_j are coincident with the 3-axis, then the fluctuating velocity field is two-dimensional and horizontally non-divergent:

$$u^*_{i,3} = 0; \quad u^*_{1,1} + u^*_{2,2} = 0, \tag{47}$$

as is the case for the Taylor-Proudman relevant to AWMR fluctuations.

Rapid rotation limit of the momentum equations. The advective derivative of the AWAR fluctuating velocity can be approximated as, equation (21),

$$\frac{D}{Dt}u_i^* \approx R_{ij}^a u_j^*. \tag{48}$$

From equation (43) one obtains

$$(R_{ij}^a + W_{ij}^*)u_j^* \approx -p_{,i}^*. \tag{49}$$

This is a "geostrophic" balance for AWAR fluctuations. Such balance of the momentum equations, as shown in §3.1 and §3.2, leads directly to a streamfunction representation for the components of the velocity in planes perpendicular to the axis of rotation. The streamfunction representation is, of course, due to the horizontal nondivergence of the flow.

The results for AWOR fluctuation can be obtained from above results by recognizing that for AWOR fluctuations $R_{ij}^a = 0$. The results for the AWMR fluctuations, the Taylor-Proudman theorem in an inertial, is now derived.

AWMR fluctuations in an inertial frame. The advection of these fluctuations in the inertial frame is given by (equation 19):

$$\frac{D}{Dt}u_i^* = \ W_{ij}^* u_j^*; \quad \frac{D}{Dt}\omega_i^* = \ W_{ij}^*\omega_j^*; \tag{50}$$

Here W_{ij}^* is the mean-flow rotation rate. This indicates that the Oldroyd derivative of these fluctuations are small in the inertial frame, consistent with the advective Lagrangian derivative being small in the rotating frame. By recognizing $R_{ij}^a = W_{ij}^*$ in equations (46) and (49), we obtain the Taylor-Proudman theorem and its corresponding geostrophic balance in an inertial reference frame:

$$W_j^* u_i^*{}_{,j} \approx 0; \quad 2W_{ij}^* u_j^* = \varepsilon_{ipj} W_p^* u_j^* \approx -p_{,i}^*. \tag{51}$$

That this result is identical to the Taylor-Proudman geostrophic balance equation (30) is seen readily by recognizing that $W_k^* = 2\Omega_k^*$.

4. Discussion. Some useful insights into the results derived are now given. Further, the implications of the results for developing turbulence models for rotating flows is also discussed.

Kinematics of the fluctuating vorticity field. The fluctuating vorticity equation indicates that all classes of fluctuating velocities are rendered two-dimensional in the rapid rotation limit (equations 47, 46). The two-dimensionality of the fluctuating velocity field has important implications. If the mean vorticity vector is oriented along the 3-axis, then the fluctuating velocity field is such that $u_{i,3} = 0$. Thus $W_3 u_{i,3} \approx 0$ for all fluctuations in the limit of rapid rotation. When this result is introduced into the vorticity evolution equation (44), we obtain

$$\frac{D}{Dt}\omega_i = W_3 u_{i,3} + \omega_j W_{ij} \approx \omega_j W_{ij}. \tag{52}$$

The fluctuating vorticity field of a two-dimensional flow undergoes a kinematic rotation in the limit of rapid rotation. The rotation rate of vorticity is equal to the mean flow rotation rate. In associated enstrophy equation is

$$\frac{D}{Dt}(\omega_i\omega_i) = \omega_i\omega_j W_{ij} = 0. \tag{53}$$

The last equality follows from the antisymmetry of W_{ij}. On the timescale rotation, the enstrophy is constant. If the fluctuating vorticity is small in comparison to the mean intrinsic vorticity at the initial time, it will continue to be small on the timescale of the intrinsic vorticity. Thus, a fluctuating velocity field that satisfies the rapid rotation limit assumptions specified in equation (6) will preserve the validity of these assumptions for all time.

The pressure-rotation balance. In a general rotating flow, the geostrophic balance between the pressure and rotation-related forces depends upon the kinematic rotation rate of the fluctuating velocity field (equation 49). Consider AWMR and AWOR fluctuations in an inertial frame, subject to identical mean-flow vorticity W^*_{ij}. The fluctuating pressure gradients for the two cases are:

(54) AWMR: $$-\,p_{,i} \approx 2W_{ij}u_j;$$

AWOR: $$-\,p_{,i} \approx W_{ij}u_j.$$

For a given intrinsic mean-vorticity, it takes twice as large a (fluctuating) pressure force to sustain AWMR velocity fluctuations, as it does to maintain AWOR velocity fluctuations. The additional force is required in the AWMR case to rotate the orientation of the velocity field.

Two-dimensional fluctuations. One well known result in the mechanics of rotating fluids, (Taylor, 1917, Lighthill, 1966, Hide, 1977) is that two-dimensional velocity fields are frame-indifferent. *Any two-dimensional fluctuating velocity field,* $u_i = u_i(x_1, x_2)$*, that is a solution of the Navier-Stokes equations in an inertial system is also a solution of the Navier-Stokes equations in a rotating frame for any arbitrary (constant) rotation rate.* The pressure fields for the two evolutions are not the same but proportional by a constant scale factor.

In the previous section it has been seen that in the limit of rapid rotation AWMR, AWOR and AWAR fluctuations become independent of the coordinate along the axis of rotation – two-dimensional. Thus AWMR, AWOR and AWAR fields are frame indifferent. The shortest proof of the material frame indifference of two-dimensional fields uses the vorticity equation. The exact fluctuating vorticity equation for arbitrary (small or large) frame or mean flow rotation rate, *modulo* viscosity, is

$$\frac{D}{Dt}\omega_i + \ u_j\omega_{i,j} = [\omega_j + \mathcal{W}_j]u_{i,j} \ + \omega_j W_{ij}. \tag{55}$$

If the flow is two-dimensional independent of coordinate along the axis of rotation then $\mathcal{W}_j u_{i,j} = \ (W_j + 2\Omega_j)u_{i,j} = \ 0$ and the vorticity equation becomes

$$\frac{D}{Dt}\omega_i + \ u_j\omega_{i,j} \ = \ \omega_j u_{i,j} \ + W_{ij}\omega_j. \tag{56}$$

The vorticity equation and the kinematic Biot-Savart relation $\nabla^2 u_i = \varepsilon_{ijk}\omega_{k,j}$ are all that are required to evolve the flow. The vorticity equation and Biot-Savart are both independent of the rotation the reference frame, Ω_j. The flow is frame-indifferent in the presence of mean vorticity.

A two-dimensional fluctuating velocity field is more easily evolved by solving the vertical vorticity equation. Consider the evolution of the 3-component of vorticity. Noting that $W_{3j} \equiv 0$ and that $\omega_j u_{3,j} \equiv 0$ for velocity fields described by a streamfunction one obtains

$$\frac{D}{Dt}\omega_3 \ + \ u_j\omega_{3,j} \ = \ 0. \tag{57}$$

As $\omega_{i,3} \equiv 0$ the equation is independent of the vertical velocity, u_3. Thus $\frac{D}{Dt}\omega_3 = 0$ is a closed nonlinear equation for the streamfunction from which the horizontal components, u_1, u_2, of the velocity are fully determined. Recall that $\omega_3 = -\nabla^2\psi$ as is readily obtained from the curl of the velocity once the velocity is expressed in terms of the streamfunction.

The third component of the velocity field, u_3, is obtained from its momentum equation. With $u_{i,3} = 0$ and $p_{,3} = 0$ the 3 momentum equation devolves into the linear equation describing the advection of a passive scalar. The passive scalar is u_3. For arbitrary frame rotation, u_3 satisfies

$$\frac{D}{Dt}u_3 + u_1 u_{3,1} + u_2 u_{3,2} = 0. \tag{58}$$

This u_3 equation is, of course, independent of the mean frame rotation.

The fact that two-dimensional velocity fields are materially frame indifferent, in the context of rotating flow systems, has been known for some time (Taylor, 1917, Hide, 1977). The utility of the principle of materially frame indifference *in the limit of two-dimensional flows* has been suggested as a type of realizability principle for turbulence models by Speziale (1981). Additional amplifications on the topic of two-dimensional frame indifference are given in Speziale (1989, 1997), Kassinos and Reynolds (1994), and Ristorcelli (1997). The frame invariance of the modeled second moment equations has been implemented for single-point turbulence closures relevant to AWMR flows of the geophysical type in Ristorcelli, Lumley and Abid (1995).

Implications for Reynolds stress closures. In the Reynolds stress closure modeling of turbulent flows, one requires a closure representation for the pressure-strain covariance, Launder, Reece and Rodi (1975). The present results have important implications for pressure-strain closures. These results describe the behavior of the Navier-Stokes equations in the limit of rapid rotation. The Reynolds stress equations, which are derived from the Navier Stokes equations, must, in the limit of rapid rotation, be consistent with these results.

In Ristorcelli *et al.* (1995) the geostrophic balance, a diagnostic relationship between pressure and velocity for AWMR flows for which Taylor-Proudman is relevant,

$$2W_{ij}u_j \approx -p_{,i}\,; \tag{59}$$

has been employed to develop a sophisticated representation for the pressure-strain. This model has demonstrated advantages over models not consistent with Taylor-Proudman results in rotating flow calculations.

One of the major results of this article is a new rotation-pressure balance of the momentum equations for general rotating flows:

$$(W_{ij} + R^a_{ij})u_j \approx -p_{,i}\,. \tag{60}$$

This result is used by Girimaji (1997) to develop a pressure-strain covariance model for mean vorticity dominated flows. The resulting pressure-strain model is the first of its class to distinguish between rotating shear and elliptic flows, and produces excellent qualitative and quantitative agreement with DNS data of elliptic and rotating flows.

Estimation of R^a_{ij}. Many important kinematic and dynamic properties of the fluctuating fields in a rapidly rotated mean flow can be obtained provided that the Reynolds stress rotation rate (R^a_{ij}), which is also the kinematic rotation rate of the fluctuating field is known. It would be desirable to specify R^a_{ij} in terms of mean velocity field variables. The kinematic rotation rate is likely to depend upon many factors: initial condition, boundary condition and production-mechanism of the fluctuations. For a homogeneous rotating flow in a fully developed state, the kinematic rotation rate is likely to depend only on the fluctuation production-mechanism, which is the mean strain rate. We now investigate the dependence of R^a_{ij} on the mean strain rate. This qualitative analysis will provide a better physical insight into the difference between the fluctuating fields of rotating shear flows and elliptic streamline flows.

From observations, it is well known that the angle between the principal axes of the mean strain and the Reynolds stresses is constant in equilibrium turbulence. The Reynolds stress anisotropy attaining a constant equilibrium value is a manifestation of the unchanging orientation between strain rate and Reynolds stresses. The equilibrium behavior is observed in flows with reference frame rotation, as well as those with mean flow vorticity. The rate of rotation of the Reynolds stresses must, then, coincide with the rotation of the strain rate:

$$\frac{D}{Dt}\langle u^*_i u^*_j\rangle \sim \frac{D}{Dt}S^*_{ij}. \tag{61}$$

- In elliptic streamline flows, the strain rate is constant in magnitude and orientation. Thus in an inertial reference frame:

  $$R^a_{ij} = 0. \tag{62}$$

  This indicates that velocity fluctuations in the elliptic mean streamline flow belong to the AWOR class.
- In rotating shear flows, the strain rate eigenvectors rotate with the mean flow when observed in an inertial reference frame. The intrinsic mean vorticity, $\mathcal{W}_{ij}$, and the rotation rate of the strain rate are equal.

  $$R^a_{ij} = \mathcal{W}_{ij} = 2\varepsilon_{ikj}\Omega_j \tag{63}$$

  where Ω_j is the reference frame rotation rate. Therefore, the velocity fluctuations in the rotating shear are likely to belong to the AWMR category.

5. Summary and Conclusion. The behavior of fluctuating motions subject to a general rapid rotation has been investigated. The forms of rotation considered include reference-frame rotation, mean-flow rotation and any arbitrary combinations of the two basic forms.

In the rapid rotation limit it has been demonstrated, §2.1, that the fluctuating velocity field advection, to leading order, can be approximated by a kinematic rotation given by

$$\frac{D}{Dt}u_i \approx R_{ij}u_j. \tag{64}$$

The kinematic rotation rate tensor R_{ij} was shown to be equal to the rotation rate of the Reynolds stress tensor. The advection of fluctuating vorticity corresponding to a kinematically rotated velocity field is given by

$$\frac{D}{Dt}\omega_i \approx \frac{1}{2}(W_j - R_j)u_{i,j} + W_{ij}\omega_j = \frac{1}{2}(W_j u_{j,i} - R_j u_{i,j}). \tag{65}$$

These two equations form the foundation of this study. The introduction of these kinematic approximations into momentum and vorticity equations indicated the behavior of the fluctuating velocity and pressure fields in the rapid rotation limit. The rotating flows were characterized according to R_{ij}, the rotation of the Reynolds stress tensor, and the behavior of each class was investigated in §3.

The results of the analysis for each class are summarized below:

- $R_{ij} = W_{ij}$: This fluctuating velocity field is advected with mean-flow rotation rate (AWMR). The AWMR class of fluctuations can be associated with reference-frame rotation flows. The vorticity and momentum equations, in this case, lead to the classical Taylor-Proudman theorem and the geostrophic balance, respectively. The Taylor-Proudman theorem results from assuming that the Lagrangian derivative, in the rotating reference frame, is small: in the rotating frame, on the time scale of the rotation, the flow is frozen. This is tantamount to assuming, in an inertial frame, that the Oldroyd derivative is small. In an inertial system, these fluctuations while rapidly varying, are merely kinematically rotated. In the limit of rapid rotation the fluctuating velocity field is two-dimensional – independent of the axis of rotation – and horizontally non-divergent. The Taylor-Proudman theorem and its consequences have substantial empirical validation: these results are essential to understanding large (synoptic) scale geophysical flows taking place in the atmosphere and the ocean. The geostrophic balance is used to produce weather maps.
- $R_{ij} = 0$: This fluctuating velocity field is advected without rotation by the mean flow (AWOR). This class of fluctuations are likely to be associated with flows which have mean-flow rotation and no reference frame rotation. For AWOR fluctuating velocity field the Lagrangian derivative, in an inertial frame, is small; in the inertial frame, on the time scale of the rotation, the flow is frozen. In the limit of rapid rotation, the vorticity equation indicates that the fluctuating velocity field is, again, two-dimensional and horizontally non-divergent. However, the momentum equation indicates a different balance between rotation and pressure forces. These results appear relevant to turbulence closures of highly vortical flows of engineering interest.
- $R_{ij} = R^a_{ij}$: This class of fluctuations will be encountered in flows which are subjected to both reference-frame and mean-flow rotations. The fluctuating velocity field is, again, two-dimensional and horizontally non-divergent. The pressure-rotation balance, however, depends on the proportion of reference-frame and mean-flow rotation.

The asymptotic limits of rapid rotation obtained in this paper are of substantial fundamental interest as can be seen from their physical meaning, as well as the kinematic and dynamic consequences. These asymptotic states of rapidly rotating flows have important implications for turbulence modeling — Speziale (1981, 1989, 1997), Kassinos and Reynolds (1994), Ristorcelli *et al.* (1995), Ristorcelli (1997). To be consistent with Navier-Stokes equations, pressure-strain models of rapidly rotating flows must satisfy the geostrophic constraint as implied by the Taylor-Proudman theorem. Ristorcelli *et al.* (1995) have employed this constraint to develop a pressure-strain covariance model for flows in rotating systems. In contradistinction, in flows subjected to large mean vorticity, pressure-strains model must be consistent with a new geostrophic constraint derived in this paper. This constraint is used by Girimaji (1997) to derive a pressure-strain model able to predict the peculiar behavior of Reynolds stresses observed in the important elliptic streamline flows.

REFERENCES

G. A. Blaisdell and K. Shariff (1996), *Simulation and modeling of elliptic streamline flow.* Proceedings of 1996 Summer Program. Center for Turbulence Research, Stanford University, Stanford, California.

S. S. Girimaji (1997), ICASE report in progress.

H. P. Greenspan (1968), *Theory of Rotating Fluids.* Cambridge U. Press.

R. Hide (1977), *Experiments in rotating fluids.* Q.J. Met. Soc. 103, p. 1.

S. C. Kassinos and W. C. Reynolds (1994), *A structure-based model for the rapid-distortion of homogeneous turbulence.* Ph.D. Thesis, Report TF-61s, Dept. of Mechanical Engineering, Stanford University, Stanford, CA.

B. E. Launder, G.J. Reece and W. Rodi (1975), *Progress in the development of a Reynolds-stress turbulence closure.* J. Fluid Mech., 68, p. 537.

J. M. Lighthill (1966), *Dynamics of rotating fluids: a survey.* J. Fluid Mech., 26, p. 411.

J. Pedloski (1982), *Geophysical Fluid Dynamics.* Springer Verlag.

J. Proudman (1916), *On the motions of solids in a liquid possessing vorticity.* Proc. Roy. Soc. Lond., A92, p. 408.

J. R. Ristorcelli (1997), *Toward a turbulence constitutive relation for geophysical flows.* ICASE report 96-66. To appear Theoret. Comput. Fluid Dynamics.

J. R. Ristorcelli, J.L. Lumley and R. Abid (1995), *A rapid-pressure covariance representation consistent with the Taylor-Proudman theorem materially-frame-indifferent in the 2D limit.* J. Fluid Mech., 292, p. 111-152.

C. G. Speziale (1981), *Some interesting properties of two-dimensional turbulence.* Phys. Fluids A, 24, p. 1425.

C. G. Speziale (1989), *Turbulence modeling in noninertial frames of reference.* Theoret. Comput. Fluid Dynamics, 1, p. 3.

C. G. Speziale (1997), *A review of material frame-indifference in mechanics.* Submitted to ASME Applied Mechanics Reviews.

G. I. Taylor (1917), *Motions of solids in fluids when the flow is not irrotational.* Proc. Roy. Soc. Lond., A104, p. 213.

D. J. Tritton (1977), *Physical Fluid Dynamics.* Van Nostrand Rheingold.