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The Variational Method for Aerodynamic Optimization Using the Navier-Stokes Equations

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THE VARIATIONAL METHOD FOR AERODYNAMIC OPTIMIZATION USING THE NAVIER-STOKES EQUATIONS*

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Abstract. This report describes the formulation of an aerodynamic shape design methodology using a compressible viscous flow model based on the Reynolds-Averaged Navier-Stokes equations. The aerodynamic shape is described by a set of geometrical design variables. The design problem is formulated as an optimization problem stated in terms of an aerodynamic objective functional which has to be minimized. The design scheme employs a gradient-based optimization algorithm in order to obtain the optimum values of the design variables. The gradient of the aerodynamic functional with respect to the design variables is computed by means of the variational method, which requires the solution of an adjoint problem. The formulation of the adjoint problem is described which leads to a set of adjoint equations and boundary conditions. Using the flow variables and the adjoint variables, an expression for the gradient has been constructed. Computational results are presented for an inverse problem of an airfoil. It will be shown that, starting from an initial geometry of the NACA 0012 airfoil, the target pressure distribution and geometry of a best-fit of the RAE 2822 airfoil in a transonic flow condition has been reconstructed successfully.

Key words. aerodynamic optimization, airfoil design, variational method, optimal control, inverse design, Navier-Stokes equations

Subject classification. Applied and Numerical Mathematics

1. Introduction. Methodologies for solving aerodynamic shape design problems can be distinguished into two classes: (i) inverse methodology and (ii) optimization methodology. The distinction is based on how the design problem is formulated.

In the inverse methodology, the design problem is posed in terms of a prescribed target pressure distribution which has to be realized on the surface of the shape. The designer is assumed to be able to prescribe the target pressure distribution in such a way that it reflects required aerodynamic characteristics like lift, drag, pitching moment, and boundary layer properties which determine the aerodynamic performance. Inverse methods assist the designer by constructing an aerodynamic shape which generates the target pressure distribution (Refs. [22], [11], [10], [5]).

In the optimization methodology, the design problem is posed as a minimization problem of an aerodynamic objective functional subject to constraints on the geometric and aerodynamic properties. Optimization methods assist the designer in locating the minimum of the objective while satisfying the constraints. From the practical point of view, aerodynamic optimization methods, pioneered by Hicks et al. [14], are more attractive since these methods can handle a large class of design problems, including those classified as inverse problems. This report describes a contribution to the development in the aerodynamic optimization methodology.

Aerodynamic optimization methods can be distinguished into two categories: (i) global methods and (ii) local methods. Global methods, such as those based on the *genetic algorithm* [9], are aimed at obtaining the

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global optimum. These methods are most useful for cases in which multiple minima are present in the design space. It is widely known, however, that global methods incur large computational effort, where hundreds or even thousands of flow analyses may be needed before the global optimum can be found.

Local methods use the information on the gradient of the objective for locating the optimum. Therefore, for cases with multiple minima, local methods are limited to produce only one of the minima (i.e., the local optimum), the actual value of which depends on the starting point of the optimization process. Because of the modest computational requirement, and since any local optimum represents an improvement over an existing design, local methods are very useful design tools. The method described in this report belongs to this category.

Recent developments in the gradient-based aerodynamic optimization methodology suggest that two main streams may be distinguished: (i) the method of sensitivity analysis and (ii) the variational method. This distinction is based on how the gradient is computed.

The formulation of the gradient using the sensitivity analysis method (Refs. [27], [24], [15]), is done on a discrete level, which means that one must deal with the discrete form of the flow equations. This method has the advantage that the sensitivities of the flow properties on the grid points can be determined. Once these become available, the gradient of an aerodynamic functional can be computed easily using the chain rule. However, the computational effort strongly depends on the number of design variables. For each design variable, a sensitivity equation in the form of a (large) linear system of equations must be solved. The computational requirement can therefore be prohibitive if a large design space is to be covered.

The formulation of the gradient using the variational method can be done either on a discrete level (Refs. [2], [23], [6], [21]) or a continuous level (Refs. [25], [18], [20], [19], [26], [1], [17], [16], [28]). This method needs the values of the so-called adjoint variables as the solution of a set of adjoint equations. The numerical solution procedure for solving the flow equations can be adopted for solving the adjoint equations. The gradient is expressed in terms of the flow variables and the adjoint variables. The computational effort for obtaining the gradient is not determined by the number of design variables. Instead, it is determined by the number of adjoint equations that must be solved, which is equal to the number of aerodynamic functionals of the aerodynamic objective and constraints. Anticipating that the number of design variables is significantly larger than the number of aerodynamic functionals, which is true in many practical cases, the variational method has a significant advantage over the method of sensitivity analysis.

This report describes the design approach utilizing the variational method for airfoil design using the compressible Reynolds-Averaged Navier-Stokes (RANS) equations. Recently, the author [28] has demonstrated the feasibility of the approach in dealing with inverse problems and constrained drag-reduction problems, where the compressible viscous flow model based on the RANS equations was used for the flow calculations. An analytical expression of the adjoint equations was formulated based on the continuous form of the aerodynamic functional and the RANS equations. In the previous work, however, considerations from the physics of the boundary layer must still be taken for obtaining an approximation of the gradient, despite the success in obtaining true viscous adjoint solutions. Although the approximation can lead to useful results, as shown in Ref. [28], it is desirable to have a gradient expression which is derived consistently using the RANS equations. The objective of the present study is therefore to obtain the true RANS-based gradient expression.

2. Statement of the Design Problem. The design problem being addressed is formulated as a minimization problem of an aerodynamic functional \mathcal{F} :

$$(2.1) \quad \text{Minimize } \mathcal{F}(\mathbf{Q}, \boldsymbol{\theta})$$

where \mathbf{Q} is the vector of flow variables, and $\boldsymbol{\theta}$ is a vector representing the geometric parameters that define the aerodynamic shape. The vector $\boldsymbol{\theta}$ is treated as the design variables, the optimal value of which is to be determined. There is an implicit dependency of \mathbf{Q} upon $\boldsymbol{\theta}$ through the RANS equations for a given onset flow condition.

Problem (2.1) is to be solved by means of an iterative gradient-based optimization algorithm. This requires information on the gradient of \mathcal{F} with respect to $\boldsymbol{\theta}$ for each iterate. An efficient way of treating the implicit dependency of \mathbf{Q} upon $\boldsymbol{\theta}$ in evaluating the gradient is through the variational method. In the variational method, an adjoint problem must be formulated in which a set of adjoint equations are to be solved subject to proper adjoint boundary conditions. The gradient is expressed in terms of the flow variables (i.e., the solution of the flow problem) and the adjoint variables (i.e., the solution of the adjoint problem).

3. The Reynolds-Averaged Navier-Stokes Equations. Assuming adiabatic flow and no external forces, the time-dependent RANS equations in the conservative form are written as

$$(3.1) \quad \frac{\partial \mathbf{Q}}{\partial t} + \vec{\nabla} \cdot \vec{\mathbf{F}} = 0 \text{ in } \Omega,$$

where Ω is the flow domain, and \mathbf{Q} is the vector of conservative time-averaged flow variables,

$$(3.2) \quad \mathbf{Q} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix},$$

which are non-dimensionalized with respect to the free stream. At the steady-state, equation (3.1) becomes

$$(3.3) \quad \vec{\nabla} \cdot \vec{\mathbf{F}} = 0.$$

The flux $\vec{\mathbf{F}}$ consists of the convective, $\vec{\mathbf{F}}_c$, and viscous, $\vec{\mathbf{F}}_v$, flux vectors,

$$(3.4) \quad \vec{\mathbf{F}} = \vec{\mathbf{F}}_c - \vec{\mathbf{F}}_v.$$

The convective flux vector is defined as

$$(3.5) \quad \vec{\mathbf{F}}_c = \begin{pmatrix} \mathbf{f}_c \\ \mathbf{g}_c \end{pmatrix},$$

where \mathbf{f}_c and \mathbf{g}_c are the Cartesian components given by

$$(3.6) \quad \mathbf{f}_c = \begin{pmatrix} \rho u \\ \rho u u + p \\ \rho u v \\ (\rho E + p)u \end{pmatrix}, \quad \mathbf{g}_c = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v v + p \\ (\rho E + p)v \end{pmatrix},$$

where ρ , u , v , p and E are the air density, x - and y -velocity components, pressure, and total energy, respectively. The viscous flux vector is defined as

$$(3.7) \quad \vec{\mathbf{F}}_v = \begin{pmatrix} \mathbf{f}_v \\ \mathbf{g}_v \end{pmatrix},$$

where \mathbf{f}_v and \mathbf{g}_v are the Cartesian components given by

$$(3.8) \quad \mathbf{f}_v = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xx}u + \tau_{xy}v - q_x \end{pmatrix}, \quad \mathbf{g}_v = \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{xy}u + \tau_{yy}v - q_y \end{pmatrix}.$$

Assuming that air behaves like a Newtonian fluid, the elements τ_{xx} , τ_{xy} , and τ_{yy} of the viscous stress tensor are expressed as

$$(3.9) \quad \tau_{xx} = l(\vec{\nabla} \cdot \vec{V}) + 2\mu \frac{\partial u}{\partial x},$$

$$(3.10) \quad \tau_{yy} = l(\vec{\nabla} \cdot \vec{V}) + 2\mu \frac{\partial v}{\partial y},$$

$$(3.11) \quad \tau_{xy} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right),$$

where l is given by the Stokes hypothesis,

$$l = -\frac{2}{3}\mu.$$

The viscosity μ consists of the dynamic viscosity μ_d and the eddy viscosity μ_t ,

$$\mu = \mu_d + \mu_t,$$

where μ_d is given in terms of the onset flow condition by the Sutherland's law,

$$\frac{\mu_d}{\mu_\infty} = \left(\frac{T}{T_\infty} \right)^{3/2} \frac{T_\infty + 110}{T + 110},$$

with T the absolute temperature, while μ_t is defined by a turbulence model which, in the present study, is based on the Baldwin and Lomax model.

The Cartesian components of the heat flux vector \mathbf{q} are defined by

$$(3.12) \quad q_x = -\kappa \frac{\partial T}{\partial x},$$

$$(3.13) \quad q_y = -\kappa \frac{\partial T}{\partial y},$$

where the thermal conductivity coefficient κ consists of the laminar part, κ_d , and turbulent part, κ_t . These are related with the viscosities through the Prandtl numbers,

$$\text{Pr}_d = c_p \frac{\mu_d}{\kappa_d},$$

$$\text{Pr}_t = c_p \frac{\mu_t}{\kappa_t},$$

with c_p the specific heat at constant pressure and h the mass specific enthalpy. The Prandtl numbers are assumed to have constant values throughout the flow, $\text{Pr}_d = 0.72$ and $\text{Pr}_t = 0.9$, respectively. The total energy E per unit mass is defined as

$$E = e + \frac{1}{2}(u^2 + v^2),$$

where e is the internal energy per unit mass. The RANS equations are closed by the equation of state of a calorically perfect gas, given as

$$(3.14) \quad p = (\gamma - 1)(\rho E - \frac{1}{2}\rho(u^2 + v^2)),$$

$$(3.15) \quad T = \frac{1}{c_v}(E - \frac{1}{2}(u^2 + v^2)),$$

where $\gamma = c_p/c_v$, with c_v the specific heat at constant volume. The heat fluxes can be written in terms of the internal energy as

$$(3.16) \quad q_x = -\gamma \frac{\mu}{\text{Pr}} \frac{\partial e}{\partial x},$$

$$(3.17) \quad q_y = -\gamma \frac{\mu}{\text{Pr}} \frac{\partial e}{\partial y},$$

where

$$\gamma \frac{\mu}{\text{Pr}} = \gamma \left(\frac{\mu_d}{\text{Pr}_d} + \frac{\mu_t}{\text{Pr}_t} \right).$$

On the airfoil surface, S_a , the no-slip and adiabatic boundary conditions are applied. The no-slip boundary condition can be expressed as

$$(3.18) \quad \vec{V} \cdot \vec{n} = 0,$$

$$(3.19) \quad \vec{V} \cdot \vec{s} = 0,$$

where \vec{V} denotes the velocity vector, while \vec{n} and \vec{s} are the unit normal and tangential vectors, respectively. The adiabatic wall boundary condition reads

$$(3.20) \quad \vec{\nabla} T \cdot \vec{n} = 0.$$

This is formulated in terms of the internal energy as

$$(3.21) \quad \vec{\nabla} e \cdot \vec{n} = 0.$$

The boundary conditions are collected into a vector \mathbf{B} as follows,

$$(3.22) \quad \mathbf{B} = \begin{pmatrix} \vec{V} \cdot \vec{n} \\ \vec{V} \cdot \vec{s} \\ \vec{\nabla} e \cdot \vec{n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

4. Formulation of the Adjoint and Gradient Equations. It is assumed that the aerodynamic functional \mathcal{F} takes the form of a surface integral over the airfoil surface S_a :

$$(4.1) \quad \mathcal{F} = \int_{S_a} \psi(p, \tau_w, \boldsymbol{\theta}) dS,$$

where ψ is an explicit function of the pressure p , the wall shear stress τ_w and the design variables $\boldsymbol{\theta}$, with

$$\tau_w = \mu \frac{\partial(\vec{V} \cdot \vec{s})}{\partial n}.$$

The functional (4.1) represents a large class of design problems, including those expressed in terms of lift, drag, and pitching moment.

As \mathbf{Q} is obtained from the steady-state RANS equations with the boundary conditions (3.22), the functional \mathcal{F} is independent of the transient state. Therefore, it is sufficient to consider the steady-state RANS equations (3.3) and the boundary conditions (3.22) in the definition of a Lagrangian \mathcal{L} as follows,

$$(4.2) \quad \mathcal{L} = \int_{S_a} \psi dS + \int_{\Omega} \boldsymbol{\lambda} \cdot (\vec{\nabla} \cdot \vec{\mathbf{F}}) d\Omega + \int_{S_a} \boldsymbol{\Upsilon} \cdot \mathbf{B} dS,$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\Upsilon}$ are the vectors of Lagrange multipliers. The Lagrange multipliers $\boldsymbol{\lambda}$, also referred to as the adjoint variables, are defined in Ω and consists of four components. The Lagrange multipliers $\boldsymbol{\Upsilon}$ is a vector with three components defined on S_a .

In order to derive the adjoint and gradient equations, one must evaluate the variation of \mathcal{L} , denoted as $\delta\mathcal{L}$, implied by the independent variations of $\boldsymbol{\lambda}$, $\boldsymbol{\Upsilon}$, \mathbf{Q} , and $\boldsymbol{\theta}$,

$$\delta\mathcal{L} = \delta\mathcal{L}_\lambda + \delta\mathcal{L}_\Upsilon + \delta\mathcal{L}_Q + \delta\mathcal{L}_\theta.$$

The notation $\delta\mathcal{L}_\lambda$ refers to the variation of $\delta\mathcal{L}$ due to the variation of $\boldsymbol{\lambda}$ while the other variables are kept fixed, and similarly for $\delta\mathcal{L}_\Upsilon$, etc. The variations $\delta\mathcal{L}_\lambda$, $\delta\mathcal{L}_\Upsilon$, and $\delta\mathcal{L}_Q$ are evaluated with $\boldsymbol{\theta}$ kept fixed. Keeping $\boldsymbol{\theta}$ fixed implies a fixed domain Ω . For the variation of $\boldsymbol{\lambda}$, $\boldsymbol{\Upsilon}$, and \mathbf{Q} with a fixed domain a prime notation is introduced as $\boldsymbol{\lambda}'$, $\boldsymbol{\Upsilon}'$, and \mathbf{Q}' , respectively.

4.1. The Adjoint Equation. The variation $\boldsymbol{\lambda}'$ contributes to $\delta\mathcal{L}$ with

$$(4.3) \quad \delta\mathcal{L}_\lambda = \int_{\Omega} \boldsymbol{\lambda}' \cdot (\vec{\nabla} \cdot \vec{\mathbf{F}}) d\Omega,$$

which is cancelled by the RANS equations (3.3). The variation $\boldsymbol{\Upsilon}'$ contributes with

$$(4.4) \quad \delta\mathcal{L}_\Upsilon = \int_{S_a} \boldsymbol{\Upsilon}' \cdot \mathbf{B} dS,$$

which vanishes because of the boundary conditions (3.22).

As the RANS equations (3.3) and the boundary conditions (3.22) are satisfied, giving $\delta\mathcal{L}_\lambda = 0$ and $\delta\mathcal{L}_\Upsilon = 0$, the variation of \mathcal{L} becomes

$$\delta\mathcal{L} = \delta\mathcal{L}_Q + \delta\mathcal{L}_\theta.$$

The adjoint equations and boundary conditions follow from the condition that the contribution from the variation \mathbf{Q}' vanishes, i.e.

$$(4.5) \quad \delta\mathcal{L}_Q = 0.$$

The domain integral in equation (4.2) can be integrated by parts to give

$$(4.6) \quad \begin{aligned} \mathcal{L} = & \int_{S_a} \psi dS - \int_{S_a} \boldsymbol{\lambda} \cdot (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) dS - \int_{S_\infty} \boldsymbol{\lambda} \cdot (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) dS \\ & - \int_{\Omega} \vec{\mathbf{F}} \cdot \vec{\nabla} \boldsymbol{\lambda} d\Omega + \int_{S_a} \boldsymbol{\Upsilon} \cdot \mathbf{B} dS. \end{aligned}$$

The variation $\delta\mathcal{L}_Q$ can be expressed as

$$\begin{aligned} \delta\mathcal{L}_Q = & \int_{S_a} \frac{\partial \psi}{\partial \mathbf{Q}} \cdot \mathbf{Q}' dS - \int_{S_a} \boldsymbol{\lambda} \cdot (\vec{\mathbf{F}}' \cdot \vec{\mathbf{n}}) dS - \int_{S_\infty} \boldsymbol{\lambda} \cdot (\vec{\mathbf{F}}' \cdot \vec{\mathbf{n}}) dS \\ & - \int_{\Omega} \vec{\mathbf{F}}' \cdot \vec{\nabla} \boldsymbol{\lambda} d\Omega + \int_{S_a} \boldsymbol{\Upsilon} \cdot \mathbf{B}' dS, \end{aligned}$$

where the notations $\vec{\mathbf{F}}'$ and \mathbf{B}' refer to the variations due to \mathbf{Q}' . The flux vector $\vec{\mathbf{F}}'$ can be split into the inviscid and the viscous part:

$$\vec{\mathbf{F}}' = \vec{\mathbf{F}}'_c - \vec{\mathbf{F}}'_v.$$

It is convenient to introduce the inviscid and viscous variations, $\delta\mathcal{I}$ and $\delta\mathcal{J}$, implied by $\vec{\mathbf{F}}'_c$ and $\vec{\mathbf{F}}'_v$, respectively, and the variation $\delta\mathcal{K}$ implied by \mathbf{B}' , such that

$$(4.7) \quad \delta\mathcal{L}_Q = \delta\mathcal{F}_Q + \delta\mathcal{I} - \delta\mathcal{J} + \delta\mathcal{K}.$$

The variation $\delta\mathcal{L}_Q$ will be obtained with the assumption that

- The variation of the viscosity, μ' , can be neglected.
- The variation of the viscous terms $\vec{\mathbf{F}}'_v$ on the far-field boundary S_∞ can be dropped.

The aerodynamic functional contributes with

$$(4.8) \quad \delta\mathcal{F}_Q = \int_{S_a} \left[\frac{\partial\psi}{\partial p} (\gamma - 1)(\rho E)' + \frac{\partial\psi}{\partial\tau_w} \tau'_w \right] dS,$$

where

$$\tau'_w = \mu \frac{\partial(\vec{V}' \cdot \vec{s})}{\partial n}.$$

The inviscid term $\delta\mathcal{I}$ can be obtained as

$$(4.9) \quad \begin{aligned} \delta\mathcal{I} = & - \int_{S_a} (\vec{\lambda} \cdot \vec{n})(\gamma - 1)(\rho E)' dS - \int_{S_a} \left(\lambda_1 + \frac{a^2}{\gamma - 1} \lambda_4 \right) \rho(\vec{V}' \cdot \vec{n}) dS \\ & - \int_{S_\infty} (C^\top \boldsymbol{\lambda}) \cdot \mathbf{Q}' dS - \int_{\Omega} (\vec{A}^\top \cdot \vec{\nabla} \boldsymbol{\lambda}) \cdot \mathbf{Q}' d\Omega, \end{aligned}$$

where \vec{A} is the Jacobian of the flux vector $\vec{\mathbf{F}}_c$ with respect to \mathbf{Q} ,

$$\vec{A} = \frac{\partial \vec{\mathbf{F}}_c}{\partial \mathbf{Q}},$$

C is the Jacobian of the normal flux defined on the boundaries

$$C = \frac{\partial(\vec{\mathbf{F}}_c \cdot \vec{n})}{\partial \mathbf{Q}} = \vec{A} \cdot \vec{n},$$

$\vec{\lambda}$ is an adjoint velocity vector with the Cartesian components λ_2 and λ_3 :

$$\vec{\lambda} = \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix}.$$

The procedure for obtaining $\delta\mathcal{J}$ is described in the appendix, with the result given as equation (A.30). The variation $\delta\mathcal{K}$ can be obtained as

$$(4.10) \quad \delta\mathcal{K} = \int_{S_a} \left[\mathcal{I}_1(\vec{V}' \cdot \vec{n}) + \mathcal{I}_2(\vec{V}' \cdot \vec{s}) + \mathcal{I}_3 \frac{a^2}{\gamma} \left(\frac{\vec{\nabla}(\rho E)' \cdot \vec{n}}{p} - \frac{\vec{\nabla} \rho' \cdot \vec{n}}{(\gamma - 1)\rho} \right) \right] dS.$$

where a is the speed of sound,

$$a = \sqrt{\gamma \frac{p}{\rho}}.$$

Substituting equation (4.8), (4.9), (4.10) and (A.30) into (4.7) leads to

$$\begin{aligned}
(4.11) \quad \delta \mathcal{L}_Q = & \int_{S_a} \left[\frac{\partial \psi}{\partial p} (\gamma - 1) (\rho E)' + \frac{\partial \psi}{\partial \tau_w} \tau'_w - (\vec{\lambda} \cdot \vec{n}) (\gamma - 1) (\rho E)' - \left(\lambda_1 + \frac{a^2}{\gamma - 1} \lambda_4 \right) \rho (\vec{V}' \cdot \vec{n}) \right. \\
& + (\vec{\lambda} \cdot \vec{n}) \tau'_n + (\vec{\lambda} \cdot \vec{s}) \tau'_w \\
& - \left(\mu \frac{\partial (\vec{\lambda} \cdot \vec{s})}{\partial s} - (\vec{\lambda} \cdot \vec{n}) l H \right) (\vec{V}' \cdot \vec{n}) - \left(l \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial s} + (\vec{\lambda} \cdot \vec{s}) \mu H \right) (\vec{V}' \cdot \vec{s}) \\
& + \lambda_4 \tau_w (\vec{V}' \cdot \vec{s}) + \lambda_4 a^2 \frac{\mu}{\text{Pr}} \left(\frac{\vec{\nabla} (\rho E)' \cdot \vec{n}}{p} - \frac{\vec{\nabla} \rho' \cdot \vec{n}}{(\gamma - 1) \rho} \right) \\
& - \left(l (\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial n} \right) (\vec{V}' \cdot \vec{n}) - \mu \left(\frac{\partial (\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial s} - H (\vec{\lambda} \cdot \vec{s}) \right) (\vec{V}' \cdot \vec{s}) \\
& - a^2 \frac{\mu}{\text{Pr}} (\vec{\nabla} \lambda_4 \cdot \vec{n}) \left(\frac{(\rho E)'}{p} - \frac{\rho'}{(\gamma - 1) \rho} \right) \\
& \left. + \Upsilon_1 (\vec{V}' \cdot \vec{n}) + \Upsilon_2 (\vec{V}' \cdot \vec{s}) + \Upsilon_3 \frac{a^2}{\gamma} \left(\frac{\vec{\nabla} (\rho E)' \cdot \vec{n}}{p} - \frac{\vec{\nabla} \rho' \cdot \vec{n}}{(\gamma - 1) \rho} \right) \right] dS \\
& - \int_{S_\infty} (C^\top \boldsymbol{\lambda}) \cdot \mathbf{Q}' dS - \int_\Omega (\vec{A}^\top \cdot \vec{\nabla} \boldsymbol{\lambda} + Y^\top \mathbf{K}) \cdot \mathbf{Q}' d\Omega,
\end{aligned}$$

where Y and \mathbf{K} are given by equations (A.28) and (A.29), respectively, H is the surface curvature, and

$$\tau'_n = (l + 2\mu) \frac{\partial (\vec{V}' \cdot \vec{n})}{\partial n}.$$

Setting the domain integral in equation (4.11) to zero leads to the adjoint equations:

$$(4.12) \quad \vec{A}^\top \cdot \vec{\nabla} \boldsymbol{\lambda} + Y^\top \mathbf{K} = 0 \quad \text{in } \Omega.$$

The surface integral over S_∞ is eliminated in the same way as that described in Ref. [28], which leads to a set of far-field characteristic-based boundary conditions for the adjoint equations.

The surface integral over S_a has to be eliminated too. The contributions from $(\vec{V}' \cdot \vec{n})$ and $(\vec{V}' \cdot \vec{s})$ are cancelled by the conditions

$$(4.13) \quad \Upsilon_1 = \left(\lambda_1 + \frac{a^2}{\gamma - 1} \lambda_4 \right) \rho + \mu \frac{\partial (\vec{\lambda} \cdot \vec{s})}{\partial s} - (\vec{\lambda} \cdot \vec{n}) l H + l (\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial n},$$

$$(4.14) \quad \Upsilon_2 = l \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial s} + (\vec{\lambda} \cdot \vec{s}) \mu H - \lambda_4 \tau_w + \mu \left(\frac{\partial (\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial s} - H (\vec{\lambda} \cdot \vec{s}) \right)$$

The terms with $(\vec{\nabla} (\rho E)' \cdot \vec{n})$ and $(\vec{\nabla} \rho' \cdot \vec{n})$ are eliminated by the relation

$$(4.15) \quad \Upsilon_3 = -\lambda_4 \left(\gamma \frac{\mu}{\text{Pr}} \right).$$

The contributions from $(\rho E)'$, τ'_w and ρ' are set equal to zero by satisfying the conditions

$$(4.16) \quad \vec{\lambda} \cdot \vec{n} = \frac{\partial \psi}{\partial p},$$

$$(4.17) \quad \vec{\lambda} \cdot \vec{s} = -\frac{\partial \psi}{\partial \tau_w},$$

$$(4.18) \quad \vec{\nabla} \lambda_4 \cdot \vec{n} = 0.$$

These may be considered as corresponding to the no-slip and adiabatic wall boundary conditions (3.22). The term with τ'_n can be eliminated by the condition

$$(4.19) \quad \vec{\lambda} \cdot \vec{n} = 0.$$

This, however, conflicts with equation (4.16). This problem is circumvented by introducing a term with τ_n ,

$$\tau_n = (l + 2\mu) \frac{\partial(\vec{V} \cdot \vec{n})}{\partial n},$$

into ψ of equation (4.1), i.e.

$$\psi = \psi(p, \tau_w, \tau_n, \boldsymbol{\theta}),$$

so that equation (4.8) is modified to

$$(4.20) \quad \delta\mathcal{F}_Q = \int_{S_a} \left[\frac{\partial\psi}{\partial p} (\gamma - 1)(\rho E)' + \frac{\partial\psi}{\partial \tau_w} \tau'_w + \frac{\partial\psi}{\partial \tau_n} \tau'_n \right] dS.$$

The associated terms in equation (4.11) are replaced by the above expression appropriately, and equation (4.19) is replaced by

$$(4.21) \quad \vec{\lambda} \cdot \vec{n} = - \frac{\partial\psi}{\partial \tau_n}.$$

This can be made compatible with equation (4.16) by imposing the condition

$$(4.22) \quad \frac{\partial\psi}{\partial \tau_n} = - \frac{\partial\psi}{\partial p}.$$

This means that for a well-posed adjoint problem, there is a restriction for the aerodynamic functional \mathcal{F} . The definition of \mathcal{F} must include a term with τ_n which satisfies equation (4.22). Restriction of the same nature was recognized in Ref. [3]. In the present study, however, equation (4.22) is proposed as a general approach to ensure the well-posedness of the adjoint problem. One should also be aware that the combination of the continuity equation and the no-slip boundary conditions dictates

$$\frac{\partial(\vec{V} \cdot \vec{n})}{\partial n} = 0,$$

implying $\tau_n = 0$, so that introducing a term with τ_n into \mathcal{F} , as suggested above, does not modify the minimization problem of \mathcal{F} .

The adjoint problem can now be summarized as follows. Equation (4.12) in Ω is to be solved subject to a proper far-field characteristic-based boundary condition on S_∞ and the near-field boundary conditions (4.16)–(4.18) on S_a . The resulting vector of adjoint variables $\boldsymbol{\lambda}$ is used for obtaining Υ_1 , Υ_2 , and Υ_3 from equations (4.13)–(4.15).

4.2. The Gradient Equation. After solving the flow and adjoint equations, providing the values of \mathbf{Q} , $\boldsymbol{\lambda}$ and $\boldsymbol{\Upsilon}$, the variation of \mathcal{L} becomes

$$(4.23) \quad \delta\mathcal{L} = \delta\mathcal{L}_\theta$$

Since $\boldsymbol{\theta}$ is a parameter that describes the shape of S_a , which is part of the flow domain boundary, the variation $\delta\boldsymbol{\theta}$ implies also a variation of the flow domain Ω . As a result of this, and recognizing that

$$\begin{aligned} \mathbf{Q} &= \mathbf{Q}(\mathbf{x}), & \mathbf{x} &\in \Omega, \\ \boldsymbol{\lambda} &= \boldsymbol{\lambda}(\mathbf{x}), & \mathbf{x} &\in \Omega, \\ \boldsymbol{\Upsilon} &= \boldsymbol{\Upsilon}(\mathbf{x}), & \mathbf{x} &\in S_a, \end{aligned}$$

the variation of Ω also implies a variation of \mathbf{Q} , $\boldsymbol{\lambda}$, and $\boldsymbol{\Upsilon}$ in the form of, respectively,

$$(4.24) \quad \delta \mathbf{Q}_\Omega = \frac{\partial \mathbf{Q}}{\partial \mathbf{x}} \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} \cdot \delta \boldsymbol{\theta} \right), \quad \mathbf{x} \in \Omega,$$

$$(4.25) \quad \delta \boldsymbol{\lambda}_\Omega = \frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{x}} \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} \cdot \delta \boldsymbol{\theta} \right), \quad \mathbf{x} \in \Omega,$$

$$(4.26) \quad \delta \boldsymbol{\Upsilon}_\Omega = \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathbf{x}} \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} \cdot \delta \boldsymbol{\theta} \right), \quad \mathbf{x} \in S_a.$$

This leads to the introduction of the notion of the *deformation velocity* $\vec{\omega}$ (Refs. [8], [13], [28]):

$$(4.27) \quad \vec{\omega}(\mathbf{x}) = \vec{\boldsymbol{\chi}} \cdot \delta \boldsymbol{\theta},$$

where

$$\vec{\boldsymbol{\chi}} = (\boldsymbol{\chi}_x \quad \boldsymbol{\chi}_y)^\top = \left(\frac{\partial x}{\partial \boldsymbol{\theta}} \quad \frac{\partial y}{\partial \boldsymbol{\theta}} \right)^\top$$

The Cartesian components of $\vec{\omega}$ are defined as

$$\begin{aligned} \omega_x &= \boldsymbol{\chi}_x \cdot \delta \boldsymbol{\theta}, \\ \omega_y &= \boldsymbol{\chi}_y \cdot \delta \boldsymbol{\theta}. \end{aligned}$$

The normal and tangential components of $\vec{\omega}$ are written as

$$\begin{aligned} \omega_n &= \boldsymbol{\chi}_n \cdot \delta \boldsymbol{\theta}, \\ \omega_s &= \boldsymbol{\chi}_s \cdot \delta \boldsymbol{\theta}, \end{aligned}$$

where

$$\begin{pmatrix} \boldsymbol{\chi}_n \\ \boldsymbol{\chi}_s \end{pmatrix} = \begin{pmatrix} n_x & n_y \\ n_y & -n_x \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi}_x \\ \boldsymbol{\chi}_y \end{pmatrix}.$$

Expressions (4.24)–(4.26) can now be written in the form

$$\begin{aligned} \frac{\partial \mathbf{Q}}{\partial \mathbf{x}} \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} \cdot \delta \boldsymbol{\theta} \right) &= \vec{\nabla} \mathbf{Q} \cdot \vec{\omega}, \quad \mathbf{x} \in \Omega, \\ \frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{x}} \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} \cdot \delta \boldsymbol{\theta} \right) &= \vec{\nabla} \boldsymbol{\lambda} \cdot \vec{\omega}, \quad \mathbf{x} \in \Omega, \\ \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathbf{x}} \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} \cdot \delta \boldsymbol{\theta} \right) &= \left(\frac{\partial \boldsymbol{\Upsilon}}{\partial s} \right) \omega_s, \quad \mathbf{x} \in S_a. \end{aligned}$$

These represent the so-called *convective variations* of \mathbf{Q} , $\boldsymbol{\lambda}$ and $\boldsymbol{\Upsilon}$, respectively. The convective variation refers to the interpretation that, the domain moves in space with the speed $\vec{\omega}$ which gives rise to the deformation of the domain boundaries.

As a complement to the convective variation, the notion *local variation* can be introduced for the variations \mathbf{Q}' , $\boldsymbol{\lambda}'$, and $\boldsymbol{\Upsilon}'$. The local and convective variations constitute the *total variation* represented by a *material derivative*:

$$\begin{aligned} \dot{\mathbf{Q}} &= \mathbf{Q}' + \vec{\nabla} \mathbf{Q} \cdot \vec{\omega}, \\ \dot{\boldsymbol{\lambda}} &= \boldsymbol{\lambda}' + \vec{\nabla} \boldsymbol{\lambda} \cdot \vec{\omega}, \\ \dot{\boldsymbol{\Upsilon}} &= \boldsymbol{\Upsilon}' + \left(\frac{\partial \boldsymbol{\Upsilon}}{\partial s} \right) \omega_s, \end{aligned}$$

where the first and second terms in the right-hand sides are the local and convective variations, respectively.

The concept of material derivative can also be applied to the variation of geometric properties, the formulae of which are given in Ref. [28] as

- The material derivative of a unit normal vector \vec{n} :

$$(4.28) \quad \dot{\vec{n}} = - \left(\frac{\partial \omega_n}{\partial s} + H \omega_s \right) \vec{s}.$$

where H is the surface curvature.

- The material derivative of a unit tangential vector \vec{s} :

$$(4.29) \quad \dot{\vec{s}} = \left(\frac{\partial \omega_n}{\partial s} + H \omega_s \right) \vec{n}.$$

- The material derivative of a surface element ds :

$$(4.30) \quad d\dot{S} = \left(\frac{\partial \omega_s}{\partial s} + H \omega_n \right) dS.$$

- The material derivative of a volume element $d\Omega$:

$$(4.31) \quad d\dot{\Omega} = (\vec{\nabla} \cdot \vec{\omega}) d\Omega.$$

For functionals of the form

$$\Phi_1 = \int_{\Omega} f d\Omega,$$

use can be made of the material derivative formula:

$$(4.32) \quad \dot{\Phi}_1 = \int_{\Omega} f' d\Omega - \int_S f \omega_n dS,$$

whereas for functionals of the form

$$\Phi_2 = \int_S f dS,$$

the material derivative can be written as

$$(4.33) \quad \dot{\Phi}_2 = \int_S \left[f' + \vec{\nabla} f \cdot \vec{\omega} + f \left(\frac{\partial \omega_s}{\partial s} + H \omega_n \right) \right] dS.$$

These material derivative formulae are applied for obtaining the variation of $\delta \mathcal{L}$. Equations (4.33) and (4.32) are applied for each surface and domain integral, respectively, which appears in the expression of \mathcal{L} . It is noted that the terms with f' in formulae (4.33) and (4.32) must be disregarded, because these refer to the local variations which have been eliminated by the solutions of the flow and adjoint problems.

The far-field boundary S_{∞} and the trailing edge can be assumed fixed with $\vec{\omega} = 0$, and the corresponding terms can be dropped. For the sake of brevity, a tilde notation is introduced for the convective variations. The variation of \mathcal{L} due to the variation of θ can be derived from equation (4.6). With the no-slip and

adiabatic wall boundary conditions (3.22) taken into account, this leads to

$$\begin{aligned}
(4.34) \quad \delta \mathcal{L} = \int_{S_a} & \left[\frac{\partial \psi}{\partial p} (\gamma - 1) \widetilde{\rho E} + \frac{\partial \psi}{\partial \tau_w} \widetilde{\tau}_w + \frac{\partial \psi}{\partial \tau_n} \widetilde{\tau}_n + \left(\frac{\partial \psi}{\partial \boldsymbol{\theta}} \right) \cdot \delta \boldsymbol{\theta} \right. \\
& + \left(\psi - (\vec{\lambda} \cdot \vec{n}) p + (\vec{\lambda} \cdot \vec{s}) \tau_w \right) \left(\frac{\partial \omega_s}{\partial s} + H \omega_n \right) \\
& + (\vec{\lambda} \cdot \vec{n}) \widetilde{\tau}_n - (\vec{\lambda} \cdot \vec{n}) (\gamma - 1) \widetilde{\rho E} - (\vec{\lambda} \cdot \vec{n} + \vec{\lambda} \cdot \dot{\vec{n}}) p \\
& + (\vec{\lambda} \cdot \vec{s}) \widetilde{\tau}_w + (\vec{\lambda} \cdot \vec{s} + \vec{\lambda} \cdot \dot{\vec{s}}) \tau_w - \left(\lambda_1 + \frac{a^2}{\gamma - 1} \lambda_4 \right) \rho (\vec{V} \cdot \vec{n}) \\
& - \left(\mu \frac{\partial (\vec{\lambda} \cdot \vec{s})}{\partial s} - (\vec{\lambda} \cdot \vec{n}) l H \right) (\vec{V} \cdot \vec{n}) - \left(l \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial s} + (\vec{\lambda} \cdot \vec{s}) \mu H \right) (\vec{V} \cdot \vec{s}) \\
& + \lambda_4 \tau_w (\vec{V} \cdot \vec{s}) + \lambda_4 \gamma \frac{\mu}{\text{Pr}} (\vec{\nabla} \tilde{e} \cdot \vec{n} + \vec{\nabla} e \cdot \dot{\vec{n}}) + (\vec{\mathbf{F}} \cdot \vec{\nabla} \boldsymbol{\lambda}) \omega_n \\
& \left. + \Upsilon_1 (\vec{V} \cdot \vec{n}) + \Upsilon_2 (\vec{V} \cdot \vec{s}) + \Upsilon_3 (\vec{\nabla} \tilde{e} \cdot \vec{n} + \vec{\nabla} e \cdot \dot{\vec{n}}) \right] dS.
\end{aligned}$$

where $\dot{\vec{n}}$ and $\dot{\vec{s}}$ are given by equation (4.28) and (4.29), respectively, while

$$\begin{aligned}
\widetilde{\rho E} &= \vec{\nabla} (\rho E) \cdot \vec{\omega}, \\
\tilde{e} &= \vec{\nabla} e \cdot \vec{\omega}, \\
\vec{V} &= \begin{pmatrix} \vec{\nabla} u \cdot \vec{\omega} \\ \vec{\nabla} v \cdot \vec{\omega} \end{pmatrix}, \\
\vec{\lambda} &= \begin{pmatrix} \vec{\nabla} \lambda_2 \cdot \vec{\omega} \\ \vec{\nabla} \lambda_3 \cdot \vec{\omega} \end{pmatrix}, \\
\widetilde{\tau}_w &= (\vec{\nabla} \vec{V} \cdot \vec{n}) \cdot \vec{s} + (\vec{\nabla} \vec{V} \cdot \dot{\vec{n}}) \cdot \vec{s} + (\vec{\nabla} \vec{V} \cdot \vec{n}) \cdot \dot{\vec{s}}, \\
\widetilde{\tau}_n &= (\vec{\nabla} \vec{V} \cdot \vec{n}) \cdot \vec{n} + (\vec{\nabla} \vec{V} \cdot \dot{\vec{n}}) \cdot \vec{n} + (\vec{\nabla} \vec{V} \cdot \vec{n}) \cdot \dot{\vec{n}}.
\end{aligned}$$

Substituting the expressions for Υ_1 , Υ_2 , and Υ_3 given by equations (4.13)–(4.15), into equation (4.34) yields

$$\begin{aligned}
\delta \mathcal{L} = \int_{S_a} & \left[\frac{\partial \psi}{\partial p} (\gamma - 1) \widetilde{\rho E} + \frac{\partial \psi}{\partial \tau_w} \widetilde{\tau}_w + \frac{\partial \psi}{\partial \tau_n} \widetilde{\tau}_n + \left(\frac{\partial \psi}{\partial \boldsymbol{\theta}} \right) \cdot \delta \boldsymbol{\theta} \right. \\
& + \left(\psi - (\vec{\lambda} \cdot \vec{n}) p + (\vec{\lambda} \cdot \vec{s}) \tau_w \right) \left(\frac{\partial \omega_s}{\partial s} + H \omega_n \right) \\
& + (\vec{\lambda} \cdot \vec{n}) \widetilde{\tau}_n - (\vec{\lambda} \cdot \vec{n}) (\gamma - 1) \widetilde{\rho E} - (\vec{\lambda} \cdot \vec{n} + \vec{\lambda} \cdot \dot{\vec{n}}) p \\
& + (\vec{\lambda} \cdot \vec{s}) \widetilde{\tau}_w + (\vec{\lambda} \cdot \vec{s} + \vec{\lambda} \cdot \dot{\vec{s}}) \tau_w + (\vec{\mathbf{F}} \cdot \vec{\nabla} \boldsymbol{\lambda}) \omega_n \\
& \left. + \left(l (\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial n} \right) (\vec{V} \cdot \vec{n}) + \mu \left(\frac{\partial (\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial (\vec{\lambda} \cdot \vec{n})}{\partial s} - H (\vec{\lambda} \cdot \vec{s}) \right) (\vec{V} \cdot \vec{s}) \right] dS.
\end{aligned}$$

The adjoint boundary conditions (4.16)–(4.18) and the condition (4.22) cancels the contributions from $\widetilde{\rho E}$,

$\tilde{\tau}_w$, and $\tilde{\tau}_n$, so that the above equation reduces to

$$\begin{aligned} \delta\mathcal{L} = \int_{S_a} \left[\left(\frac{\partial\psi}{\partial\boldsymbol{\theta}} \right) \cdot \delta\boldsymbol{\theta} + \left(\psi - (\vec{\lambda} \cdot \vec{n})p + (\vec{\lambda} \cdot \vec{s})\tau_w \right) \left(\frac{\partial\omega_s}{\partial s} + H\omega_n \right) \right. \\ \left. - (\vec{\lambda} \cdot \vec{n} + \vec{\lambda} \cdot \dot{\vec{n}})p + (\vec{\lambda} \cdot \vec{s} + \vec{\lambda} \cdot \dot{\vec{s}})\tau_w + (\vec{\mathbf{F}} \cdot \vec{\nabla}\boldsymbol{\lambda})\omega_n \right. \\ \left. + \left(l(\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial n} \right) (\vec{V} \cdot \vec{n}) + \mu \left(\frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} - H(\vec{\lambda} \cdot \vec{s}) \right) (\vec{V} \cdot \vec{s}) \right] dS. \end{aligned}$$

Expanding $\tilde{\lambda}$, \tilde{V} and $\vec{\mathbf{F}} \cdot \vec{\nabla}\boldsymbol{\lambda}$ gives

$$\begin{aligned} \delta\mathcal{L} = \int_{S_a} \left[\left(\frac{\partial\psi}{\partial\boldsymbol{\theta}} \right) \cdot \delta\boldsymbol{\theta} + \left(\psi - (\vec{\lambda} \cdot \vec{n})p + (\vec{\lambda} \cdot \vec{s})\tau_w \right) \left(\frac{\partial\omega_s}{\partial s} + H\omega_n \right) \right. \\ \left. - \left((\vec{\nabla}\lambda_2 \cdot \vec{\omega}) n_x + (\vec{\nabla}\lambda_3 \cdot \vec{\omega}) n_y - \left(\frac{\partial\omega_n}{\partial s} + H\omega_s \right) (\vec{\lambda} \cdot \vec{s}) \right) p \right. \\ \left. + \left((\vec{\nabla}\lambda_2 \cdot \vec{\omega}) n_y - (\vec{\nabla}\lambda_3 \cdot \vec{\omega}) n_x + \left(\frac{\partial\omega_n}{\partial s} + H\omega_s \right) (\vec{\lambda} \cdot \vec{n}) \right) \tau_w \right. \\ \left. - \left(\tau_{xx} \frac{\partial\lambda_2}{\partial x} + \tau_{xy} \left(\frac{\partial\lambda_3}{\partial x} + \frac{\partial\lambda_2}{\partial y} \right) + \tau_{yy} \frac{\partial\lambda_3}{\partial y} - \vec{q} \cdot \vec{\nabla}\lambda_4 \right) \omega_n \right. \\ \left. + \left(l(\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial n} \right) \left((\vec{\nabla}u \cdot \vec{\omega}) n_x + (\vec{\nabla}v \cdot \vec{\omega}) n_y \right) \right. \\ \left. + \mu \left(\frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} - H(\vec{\lambda} \cdot \vec{s}) \right) \left((\vec{\nabla}u \cdot \vec{\omega}) n_y - (\vec{\nabla}v \cdot \vec{\omega}) n_x \right) \right] dS. \end{aligned}$$

The gradient of the aerodynamic functional \mathcal{F} with respect to the design variables $\boldsymbol{\theta}$ can be obtained from

$$\frac{d\mathcal{F}}{d\boldsymbol{\theta}} = \lim_{\delta\boldsymbol{\theta} \rightarrow 0} \frac{\delta\mathcal{L}}{\delta\boldsymbol{\theta}},$$

which can be elaborated by using the definition of $\vec{\chi}$, equation (4.27), and equations (A.11)–(A.13) to give

$$\begin{aligned} (4.35) \quad \frac{d\mathcal{F}}{d\boldsymbol{\theta}} = \int_{S_a} \left[\left(\frac{\partial\psi}{\partial\boldsymbol{\theta}} \right) + \left(\psi - (\vec{\lambda} \cdot \vec{n})p + (\vec{\lambda} \cdot \vec{s})\tau_w \right) \left(\frac{\partial\chi_s}{\partial s} + H\chi_n \right) \right. \\ \left. - \left((\vec{\nabla}\lambda_2 \cdot \vec{\chi}) n_x + (\vec{\nabla}\lambda_3 \cdot \vec{\chi}) n_y - \left(\frac{\partial\chi_n}{\partial s} + H\chi_s \right) (\vec{\lambda} \cdot \vec{s}) \right) p \right. \\ \left. + \left((\vec{\nabla}\lambda_2 \cdot \vec{\chi}) n_y - (\vec{\nabla}\lambda_3 \cdot \vec{\chi}) n_x + \left(\frac{\partial\chi_n}{\partial s} + H\chi_s \right) (\vec{\lambda} \cdot \vec{n}) \right) \tau_w \right. \\ \left. - \left(2 n_x n_y \left(\frac{\partial\lambda_2}{\partial x} - \frac{\partial\lambda_3}{\partial y} \right) + (n_y^2 - n_x^2) \left(\frac{\partial\lambda_3}{\partial x} + \frac{\partial\lambda_2}{\partial y} \right) \right) \tau_w \chi_n \right. \\ \left. - (\vec{q} \cdot \vec{\nabla}\lambda_4)\chi_n + \left(l(\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial n} \right) \left((\vec{\nabla}u \cdot \vec{\chi}) n_x + (\vec{\nabla}v \cdot \vec{\chi}) n_y \right) \right. \\ \left. + \mu \left(\frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} - H(\vec{\lambda} \cdot \vec{s}) \right) \left((\vec{\nabla}u \cdot \vec{\chi}) n_y - (\vec{\nabla}v \cdot \vec{\chi}) n_x \right) \right] dS. \end{aligned}$$

5. Numerical Procedure and Computational Results. The RANS equations (3.3) are solved by means of HI-TASK [12], [7] with the Baldwin-Lomax turbulence model [4] implemented. The numerical procedure in HI-TASK deals with the time-dependent RANS equations which are integrated explicitly using a five-stage Runge-Kutta scheme towards the steady-state. The spatial discretization employs a cell-vertex finite-volume scheme. Jameson’s type of artificial dissipation is introduced consisting of 2-nd order and 4-th order terms. The convergence towards the steady-state is accelerated by means of a multigrid procedure. Characteristic-based boundary conditions are applied on S_∞ .

The adjoint solver employs a similar numerical scheme as that used in the flow solver. The procedure deals with the time-dependent form of the adjoint equations which are integrated explicitly towards the steady-state using the same five-stage Runge-Kutta scheme. Artificial dissipation is introduced for the adjoint equations in the same way as that for the flow equations. The convergence towards the steady-state is accelerated by the same multigrid procedure. Characteristic-based boundary conditions are also applied on S_∞ (Ref. [28]).

The optimization routine FSQP [29] and the flow solver HI-TASK are integrated with the adjoint solver and the gradient evaluator, which forms the design code. A design test case is defined representing a reconstruction-type inverse problem. The target pressure coefficient C_p is obtained from a flow analysis of a best-fit of the RAE 2822 airfoil with the flow condition:

$$M = 0.73, \quad \alpha = 2^\circ, \quad \text{Re} = 6.5 \times 10^6,$$

where M , α and Re are the Mach number, angle of attack, and Reynolds number, respectively. The target C_p distribution is defined on the airfoil chord with proper distinction between the lower and upper surface of the airfoil. The NACA 0012 airfoil is used as the starting airfoil geometry.

The objective functional to be minimized has the form

$$\mathcal{F} = \frac{1}{2} \int_0^1 (C_p - C_{p,t})_l^2 dx + \frac{1}{2} \int_0^1 (C_p - C_{p,t})_u^2 dx$$

where x is coincident with the airfoil chord, and $C_{p,t}$ is the target value. The subscripts l and u refer to the lower and upper surface, respectively. \mathcal{F} can also be expressed in the form

$$\mathcal{F} = \frac{1}{2} \int_{S_a} (C_p - C_{p,t})^2 |n_y| dS.$$

The function ψ in this case is defined as

$$\psi = \frac{1}{2} (C_p - C_{p,t})^2 |n_y|.$$

It is noted that

$$C_p = 2(p - p_\infty),$$

with p, p_∞ non-dimensionalized by $\rho_\infty V_\infty^2$. For this case,

$$\frac{\partial \psi}{\partial p} = 2(C_p - C_{p,t}) |n_y|$$

Equation (4.22) requires that \mathcal{F} must be modified to

$$\mathcal{F} = \frac{1}{2} \int_{S_a} (C_p - C_{p,t})^2 |n_y| dS - 2 \int_{S_a} (C_p - C_{p,t}) |n_y| \tau_n dS.$$

The adjoint equation (4.12) is solved subject to the adjoint boundary conditions (4.16)–(4.18), which in this case are

$$\begin{aligned}\vec{\lambda} \cdot \vec{n} &= 2(C_p - C_{p,t})|n_y|, \\ \vec{\lambda} \cdot \vec{s} &= 0, \\ \vec{\nabla} \lambda_4 \cdot \vec{n} &= 0.\end{aligned}$$

In the present study, the deformation velocity has been formulated as follows,

$$(5.1) \quad \vec{\omega}_x = 0 \text{ on } S_a,$$

while ω_y is defined by the curvature-continuous shape parameterization scheme described in Ref. [28]. This parameterization scheme has proved to be effective in covering a large variation of airfoil shape and, after a proper scaling, has shown to imply an efficient optimization process.

One purpose of selecting this test case is to investigate the accuracy level of the computed gradient, because the optimal solution (i.e., the target airfoil) is known beforehand. The computed gradient is considered to be of sufficient accuracy if the optimal solution can be obtained. The optimal solution is assumed to be obtained if $\mathcal{F} \leq 10^{-4}$. This means that the difference between the actual and target C_p distributions is roughly within 0.01 (engineering accuracy).

Figure 5.1 shows the design iteration history. The engineering accuracy has been achieved with 16 flow analyses. The optimization was stopped after the maximum allowable number of flow analyses (25 analyses) was exceeded. The C_p distributions and airfoils are shown in Figure 5.2. The dashed line (the optimization result) and the solid line (the target) are almost coincident, which demonstrates that the best-fit of the RAE 2822 has been closely reconstructed.

6. Conclusion. The objective of the present study is to construct an aerodynamic design methodology using the variational method in two-dimensional compressible viscous flow governed by the Reynolds-Averaged Navier-Stokes equations. The focus of the study is to obtain a correct gradient expression.

The present method has been successfully applied for solving a reconstruction-type inverse problem in a transonic flow condition. This means that the correct adjoint formulation and gradient expression have been obtained.

The numerical result presented also strongly indicates that the present method is capable of dealing with other types of design problems, as long as the adjoint problem can be formulated properly as described in the preceding sections. It is therefore suggested that the present method is applied to more practical design cases, such as those involving the criteria on lift, drag, and pitching moment.

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Appendix A. The variations considered here are the variations of the viscous terms due only to the variation \mathbf{Q}' with the assumption that

- The variation of μ with respect to the variation of \mathbf{Q} is neglected.
- The viscous terms on the far-field boundary are dropped.

The viscous term is defined as

$$\mathcal{J} = \int_{\Omega} \boldsymbol{\lambda} \cdot (\vec{\nabla} \cdot \vec{\mathbf{F}}_v) d\Omega.$$

The variation of \mathcal{J} due to \mathbf{Q}' can be expressed as

$$\delta\mathcal{J} = \int_{\Omega} \boldsymbol{\lambda} \cdot (\vec{\nabla} \cdot \vec{\mathbf{F}}'_v) d\Omega.$$

Integration by parts yields

$$(A.1) \quad \delta\mathcal{J} = - \int_{S_a} \boldsymbol{\lambda} \cdot (\vec{\mathbf{F}}'_v \cdot \vec{n}) dS - \int_{\Omega} \vec{\mathbf{F}}'_v \cdot \vec{\nabla} \boldsymbol{\lambda} d\Omega.$$

It is noted that the unit normal vector points toward the flow domain. Recalling equation (3.8) and introducing $\delta\mathcal{J}_1$, $\delta\mathcal{J}_2$, $\delta\mathcal{J}_3$, and $\delta\mathcal{J}_4$ as follows,

$$(A.2) \quad \delta\mathcal{J}_1 = \int_{S_a} \left[\lambda_2 (n_x \tau'_{xx} + n_y \tau'_{xy}) + \lambda_3 (n_x \tau'_{xy} + n_y \tau'_{yy}) \right] dS,$$

$$(A.3) \quad \delta\mathcal{J}_2 = \int_{S_a} \lambda_4 \left[n_x (\tau'_{xx} u + \tau'_{xy} v + \tau_{xx} u' + \tau_{xy} v' - q'_x) \right. \\ \left. + n_y (\tau'_{xy} u + \tau'_{yy} v + \tau_{xy} u' + \tau_{yy} v' - q'_y) \right] dS,$$

$$(A.4) \quad \delta\mathcal{J}_3 = \int_{\Omega} \left[\tau'_{xx} \frac{\partial \lambda_2}{\partial x} + \tau'_{xy} \frac{\partial \lambda_2}{\partial y} + \tau'_{xy} \frac{\partial \lambda_3}{\partial x} + \tau'_{yy} \frac{\partial \lambda_3}{\partial y} \right] d\Omega,$$

$$(A.5) \quad \delta\mathcal{J}_4 = \int_{\Omega} \left[(\tau'_{xx} u + \tau_{xx} u' + \tau'_{xy} v + \tau_{xy} v' - q'_x) \frac{\partial \lambda_4}{\partial x} \right. \\ \left. + (\tau'_{xy} u + \tau_{xy} u' + \tau'_{yy} v + \tau_{yy} v' - q'_y) \frac{\partial \lambda_4}{\partial y} \right] d\Omega.$$

equation (A.1) can be written as

$$(A.6) \quad \delta\mathcal{J} = -\delta\mathcal{J}_1 - \delta\mathcal{J}_2 - \delta\mathcal{J}_3 - \delta\mathcal{J}_4.$$

The variation $\delta\mathcal{J}_1$ will be dealt with first. Introducing a local coordinate system (n, s) , where n and s are coincident with the local normal and tangential direction on the surface, respectively, the component of the viscous stress tensor can be written as

$$(A.7) \quad \tau_{xx} = l \left(\frac{\partial(\vec{V} \cdot \vec{n})}{\partial n} + \frac{\partial(\vec{V} \cdot \vec{s})}{\partial s} - H(\vec{V} \cdot \vec{n}) \right) + 2\mu n_y H(n_y(\vec{V} \cdot \vec{n}) - n_x(\vec{V} \cdot \vec{s})) \\ + 2\mu \left(n_x^2 \frac{\partial(\vec{V} \cdot \vec{n})}{\partial n} + n_y^2 \frac{\partial(\vec{V} \cdot \vec{s})}{\partial s} \right) + 2\mu n_x n_y \left(\frac{\partial(\vec{V} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{V} \cdot \vec{n})}{\partial s} \right)$$

$$(A.8) \quad \tau_{yy} = l \left(\frac{\partial(\vec{V} \cdot \vec{n})}{\partial n} + \frac{\partial(\vec{V} \cdot \vec{s})}{\partial s} - H(\vec{V} \cdot \vec{n}) \right) + 2\mu n_x H(n_x(\vec{V} \cdot \vec{n}) + n_y(\vec{V} \cdot \vec{s})) \\ + 2\mu \left(n_y^2 \frac{\partial(\vec{V} \cdot \vec{n})}{\partial n} + n_x^2 \frac{\partial(\vec{V} \cdot \vec{s})}{\partial s} \right) - 2\mu n_x n_y \left(\frac{\partial(\vec{V} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{V} \cdot \vec{n})}{\partial s} \right)$$

$$(A.9) \quad \tau_{xy} = 2\mu n_x n_y \left(\frac{\partial(\vec{V} \cdot \vec{n})}{\partial n} - \frac{\partial(\vec{V} \cdot \vec{s})}{\partial s} \right) + \mu(n_y^2 - n_x^2) \left(\frac{\partial(\vec{V} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{V} \cdot \vec{n})}{\partial s} \right) \\ - 2\mu n_x n_y H(\vec{V} \cdot \vec{n}) - \mu(n_y^2 - n_x^2) H(\vec{V} \cdot \vec{s}).$$

where H is the surface curvature. The variation $\delta\mathcal{J}_1$ can be worked out by using the expression for τ_{xx} , τ_{yy} , and τ_{xy} given above. After some algebraic manipulation, one obtains

$$\delta\mathcal{J}_1 = \int_{S_a} \left[(\vec{\lambda} \cdot \vec{n}) \left((l + 2\mu) \frac{\partial(\vec{V}' \cdot \vec{n})}{\partial n} + l \frac{\partial(\vec{V}' \cdot \vec{s})}{\partial s} \right) + (\vec{\lambda} \cdot \vec{s}) \mu \left(\frac{\partial(\vec{V}' \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{V}' \cdot \vec{n})}{\partial s} \right) \right. \\ \left. + (\vec{\lambda} \cdot \vec{n}) l H(\vec{V}' \cdot \vec{n}) - (\vec{\lambda} \cdot \vec{s}) \mu H(\vec{V}' \cdot \vec{s}) \right] dS,$$

where $\vec{\lambda}$ is an adjoint velocity vector with λ_2 and λ_3 being the Cartesian components. The terms with the tangential derivative can be worked out with integration by parts to yield

$$\begin{aligned} \delta\mathcal{J}_1 &= (\vec{\lambda} \cdot \vec{n})l(\vec{V}' \cdot \vec{s})\Big|_l^u + (\vec{\lambda} \cdot \vec{s})\mu(\vec{V}' \cdot \vec{n})\Big|_l^u \\ &+ \int_{S_a} \left[(\vec{\lambda} \cdot \vec{n})\tau'_n + (\vec{\lambda} \cdot \vec{s})\tau'_w - \left(\mu \frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial s} - (\vec{\lambda} \cdot \vec{n})lH \right) (\vec{V}' \cdot \vec{n}) \right. \\ &\quad \left. - \left(l \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} + (\vec{\lambda} \cdot \vec{s})\mu H \right) (\vec{V}' \cdot \vec{s}) \right] dS, \end{aligned}$$

where

$$\begin{aligned} \tau'_w &= \mu \frac{\partial(\vec{V}' \cdot \vec{s})}{\partial n}, \\ \tau'_n &= (l + 2\mu) \frac{\partial(\vec{V}' \cdot \vec{n})}{\partial n}. \end{aligned}$$

while u and l refers to the upper and lower trailing edge, respectively. If the surface S_a is assumed smooth (i.e., a sharp trailing edge is assumed to have a large, but finite, curvature), and $\vec{\lambda}$ as well as \vec{V} are continuous across the trailing edge, the first two terms on the right-hand-side vanish, such that

$$\begin{aligned} \text{(A.10)} \quad \delta\mathcal{J}_1 &= \int_{S_a} \left[(\vec{\lambda} \cdot \vec{n})\tau'_n + (\vec{\lambda} \cdot \vec{s})\tau'_w - \left(\mu \frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial s} - (\vec{\lambda} \cdot \vec{n})lH \right) (\vec{V}' \cdot \vec{n}) \right. \\ &\quad \left. - \left(l \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} + (\vec{\lambda} \cdot \vec{s})\mu H \right) (\vec{V}' \cdot \vec{s}) \right] dS. \end{aligned}$$

To obtain the expression for $\delta\mathcal{J}_2$, use is made of the no-slip boundary conditions (3.18)–(3.19) which imply

$$\begin{aligned} \frac{\partial(\vec{V} \cdot \vec{n})}{\partial n} &= 0, \\ \frac{\partial(\vec{V} \cdot \vec{s})}{\partial s} &= 0, \\ \frac{\partial(\vec{V} \cdot \vec{n})}{\partial s} &= 0. \end{aligned}$$

It is noted that the first equation above is identical with the continuity equation taken on the airfoil surface with $u, v = 0$. Equations (A.7)–(A.9) can now be written as

$$\text{(A.11)} \quad \tau_{xx} = 2 n_x n_y \tau_w,$$

$$\text{(A.12)} \quad \tau_{yy} = -2 n_x n_y \tau_w,$$

$$\text{(A.13)} \quad \tau_{xy} = (n_y^2 - n_x^2)\tau_w.$$

Substituting these and equations (3.12)–(3.13) into (A.3) gives

$$\delta\mathcal{J}_2 = \int_{S_a} \lambda_4 \left(\tau_w(\vec{V}' \cdot \vec{s}) + \gamma \frac{\mu}{\text{Pr}} \vec{\nabla} e' \cdot \vec{n} \right) dS.$$

Noting that

$$\text{(A.14)} \quad \gamma e' = \frac{a^2}{\gamma - 1} \left(\frac{p'}{p} - \frac{\rho'}{\rho} \right),$$

or in terms of the total energy,

$$(A.15) \quad \gamma e' = a^2 \left(\frac{(\rho E)'}{p} - \frac{\rho'}{(\gamma - 1)\rho} \right),$$

the variation $\delta\mathcal{J}_2$ can be written as

$$(A.16) \quad \delta\mathcal{J}_2 = \int_{S_a} \lambda_4 \left[\tau_w(\vec{V}' \cdot \vec{s}) + a^2 \frac{\mu}{\text{Pr}} \left(\frac{\vec{\nabla}(\rho E)' \cdot \vec{n}}{p} - \frac{\vec{\nabla}\rho' \cdot \vec{n}}{(\gamma - 1)\rho} \right) \right] dS.$$

Next, the variation of $\delta\mathcal{J}_3$ is obtained as follows. Equations (3.9)–(3.10) are substituted into equation (A.4) which yields

$$\begin{aligned} \delta\mathcal{J}_3 = \int_{\Omega} \left[\left(\frac{\partial\lambda_2}{\partial x} + \frac{\partial\lambda_3}{\partial y} \right) l(\vec{\nabla} \cdot \vec{V}') \right. \\ \left. + 2\mu \left(\frac{\partial u'}{\partial x} \frac{\partial\lambda_2}{\partial x} + \frac{\partial v'}{\partial y} \frac{\partial\lambda_3}{\partial y} \right) + \mu \left(\frac{\partial\lambda_2}{\partial y} + \frac{\partial\lambda_3}{\partial x} \right) \left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y} \right) \right] d\Omega. \end{aligned}$$

Rearranging the coefficients of μ gives

$$\begin{aligned} \delta\mathcal{J}_3 = \int_{\Omega} \left[\left(\frac{\partial\lambda_2}{\partial x} + \frac{\partial\lambda_3}{\partial y} \right) l(\vec{\nabla} \cdot \vec{V}') \right. \\ \left. + \mu \left(\frac{\partial\lambda_2}{\partial x} \frac{\partial u'}{\partial x} + \frac{\partial\lambda_2}{\partial y} \frac{\partial u'}{\partial y} \right) + \mu \left(\frac{\partial\lambda_3}{\partial x} \frac{\partial v'}{\partial x} + \frac{\partial\lambda_3}{\partial y} \frac{\partial v'}{\partial y} \right) \right. \\ \left. + \mu \left(\frac{\partial\lambda_2}{\partial x} \frac{\partial u'}{\partial x} + \frac{\partial\lambda_3}{\partial x} \frac{\partial u'}{\partial y} \right) + \mu \left(\frac{\partial\lambda_2}{\partial x} \frac{\partial v'}{\partial x} + \frac{\partial\lambda_3}{\partial y} \frac{\partial v'}{\partial y} \right) \right] d\Omega. \end{aligned}$$

This can be written in a compact form as

$$\delta\mathcal{J}_3 = \int_{\Omega} \left[(\vec{\nabla} \cdot \vec{\lambda}) l(\vec{\nabla} \cdot \vec{V}') + \mu \left(\vec{\nabla}\lambda_2 \cdot \vec{\nabla}u' + \vec{\nabla}\lambda_3 \cdot \vec{\nabla}v' + \frac{\partial\vec{\lambda}}{\partial x} \cdot \vec{\nabla}u' + \frac{\partial\vec{\lambda}}{\partial y} \cdot \vec{\nabla}v' \right) \right] d\Omega.$$

Using the following vector identities,

$$\begin{aligned} (\vec{\nabla} \cdot \vec{\lambda}) l(\vec{\nabla} \cdot \vec{V}') &= \vec{\nabla} \cdot (l(\vec{\nabla} \cdot \vec{\lambda})\vec{V}') - \vec{\nabla}(l(\vec{\nabla} \cdot \vec{\lambda})) \cdot \vec{V}', \\ \mu \vec{\nabla}\lambda_2 \cdot \vec{\nabla}u' &= \vec{\nabla} \cdot \mu u' \vec{\nabla}\lambda_2 - u' (\vec{\nabla} \cdot \mu \vec{\nabla}\lambda_2), \\ \mu \vec{\nabla}\lambda_3 \cdot \vec{\nabla}v' &= \vec{\nabla} \cdot \mu v' \vec{\nabla}\lambda_3 - v' (\vec{\nabla} \cdot \mu \vec{\nabla}\lambda_3), \\ \mu \frac{\partial\vec{\lambda}}{\partial x} \cdot \vec{\nabla}u' &= \vec{\nabla} \cdot \mu u' \frac{\partial\vec{\lambda}}{\partial x} - u' \left(\vec{\nabla} \cdot \mu \frac{\partial\vec{\lambda}}{\partial x} \right), \\ \mu \frac{\partial\vec{\lambda}}{\partial y} \cdot \vec{\nabla}v' &= \vec{\nabla} \cdot \mu v' \frac{\partial\vec{\lambda}}{\partial y} - v' \left(\vec{\nabla} \cdot \mu \frac{\partial\vec{\lambda}}{\partial y} \right), \end{aligned}$$

one obtains

$$(A.17) \quad \begin{aligned} \delta\mathcal{J}_3 = \int_{\Omega} \left[\vec{\nabla} \cdot (l(\vec{\nabla} \cdot \vec{\lambda})\vec{V}') + \vec{\nabla} \cdot \mu u' \vec{\nabla}\lambda_2 + \vec{\nabla} \cdot \mu v' \vec{\nabla}\lambda_3 + \vec{\nabla} \cdot \mu u' \frac{\partial\vec{\lambda}}{\partial x} + \vec{\nabla} \cdot \mu v' \frac{\partial\vec{\lambda}}{\partial y} \right] d\Omega \\ - \int_{\Omega} \left[\left(\frac{\partial(l(\vec{\nabla} \cdot \vec{\lambda}))}{\partial x} + \vec{\nabla} \cdot \mu \vec{\nabla}\lambda_2 + \vec{\nabla} \cdot \mu \frac{\partial\vec{\lambda}}{\partial x} \right) u' \right. \\ \left. + \left(\frac{\partial(l(\vec{\nabla} \cdot \vec{\lambda}))}{\partial y} + \vec{\nabla} \cdot \mu \vec{\nabla}\lambda_3 + \vec{\nabla} \cdot \mu \frac{\partial\vec{\lambda}}{\partial y} \right) v' \right] d\Omega. \end{aligned}$$

The terms having the divergence form, which are collected under the first integral, are worked out using the Gauss theorem (with the integral over S_∞ neglected):

$$\begin{aligned}
\int_{\Omega} \vec{\nabla} \cdot (l(\vec{\nabla} \cdot \vec{\lambda}) \vec{V}') d\Omega &= - \int_{S_a} l(\vec{\nabla} \cdot \vec{\lambda}) (\vec{V}' \cdot \vec{n}) dS, \\
\int_{\Omega} \vec{\nabla} \cdot \mu u' \vec{\nabla} \lambda_2 d\Omega &= - \int_{S_a} \mu u' \vec{\nabla} \lambda_2 \cdot \vec{n} dS, \\
\int_{\Omega} \vec{\nabla} \cdot \mu v' \vec{\nabla} \lambda_3 d\Omega &= - \int_{S_a} \mu v' \vec{\nabla} \lambda_3 \cdot \vec{n} dS, \\
\int_{\Omega} \vec{\nabla} \cdot \mu u' \frac{\partial \vec{\lambda}}{\partial x} d\Omega &= - \int_{S_a} \mu u' \frac{\partial \vec{\lambda}}{\partial x} \cdot \vec{n} dS, \\
\int_{\Omega} \vec{\nabla} \cdot \mu v' \frac{\partial \vec{\lambda}}{\partial y} d\Omega &= - \int_{S_a} \mu v' \frac{\partial \vec{\lambda}}{\partial y} \cdot \vec{n} dS.
\end{aligned}$$

Now, the following notations are introduced

$$(A.18) \quad \Gamma_{xx} = l \vec{\nabla} \cdot \vec{\lambda} + 2\mu \frac{\partial \lambda_2}{\partial x},$$

$$(A.19) \quad \Gamma_{yy} = l \vec{\nabla} \cdot \vec{\lambda} + 2\mu \frac{\partial \lambda_3}{\partial y},$$

$$(A.20) \quad \Gamma_{xy} = \mu \left(\frac{\partial \lambda_3}{\partial x} + \frac{\partial \lambda_2}{\partial y} \right).$$

which may be considered as the elements of an adjoint stress tensor because of the close resemblance with τ_{xx} , τ_{xy} , and τ_{yy} . After some manipulations, equation (A.17) can be written as

$$\begin{aligned}
(A.21) \quad \delta \mathcal{J}_3 &= - \int_{S_a} \left[\left(l(\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial n} \right) (\vec{V}' \cdot \vec{n}) \right. \\
&\quad \left. + \mu \left(\frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} - H(\vec{\lambda} \cdot \vec{s}) \right) (\vec{V}' \cdot \vec{s}) \right] dS \\
&\quad - \int_{\Omega} \left[\left(\frac{\partial \Gamma_{xx}}{\partial x} + \frac{\partial \Gamma_{xy}}{\partial y} \right) u' + \left(\frac{\partial \Gamma_{xy}}{\partial x} + \frac{\partial \Gamma_{yy}}{\partial y} \right) v' \right] d\Omega.
\end{aligned}$$

The variation $\delta \mathcal{J}_4$ (A.5) is worked out using equations (3.9)–(3.17). This gives

$$\begin{aligned}
\delta \mathcal{J}_4 &= \int_{\Omega} \left[\left(u \frac{\partial \lambda_4}{\partial x} + v \frac{\partial \lambda_4}{\partial y} \right) l(\vec{\nabla} \cdot \vec{V}') \right. \\
&\quad + 2\mu \left(u \frac{\partial u'}{\partial x} \frac{\partial \lambda_4}{\partial x} + v \frac{\partial v'}{\partial y} \frac{\partial \lambda_4}{\partial y} \right) + \mu \left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y} \right) \left(v \frac{\partial \lambda_4}{\partial x} + u \frac{\partial \lambda_4}{\partial y} \right) \\
&\quad + \left(\tau_{xx} \frac{\partial \lambda_4}{\partial x} + \tau_{xy} \frac{\partial \lambda_4}{\partial y} \right) u' + \left(\tau_{xy} \frac{\partial \lambda_4}{\partial x} + \tau_{yy} \frac{\partial \lambda_4}{\partial y} \right) v' \\
&\quad \left. + \gamma \frac{\mu}{\text{Pr}} \left(\frac{\partial e'}{\partial x} \frac{\partial \lambda_4}{\partial x} + \frac{\partial e'}{\partial y} \frac{\partial \lambda_4}{\partial y} \right) \right] d\Omega.
\end{aligned}$$

After some manipulations, one obtains

$$\begin{aligned}
\delta \mathcal{J}_4 = \int_{\Omega} & \left[(\vec{V} \cdot \vec{\nabla} \lambda_4) l(\vec{\nabla} \cdot \vec{V}') \right. \\
& + \mu u \left(\frac{\partial u'}{\partial x} \frac{\partial \lambda_4}{\partial x} + \frac{\partial u'}{\partial y} \frac{\partial \lambda_4}{\partial y} \right) + \mu v \left(\frac{\partial v'}{\partial x} \frac{\partial \lambda_4}{\partial x} + \frac{\partial v'}{\partial y} \frac{\partial \lambda_4}{\partial y} \right) \\
& + \mu \frac{\partial \lambda_4}{\partial x} \left(u \frac{\partial u'}{\partial x} + v \frac{\partial u'}{\partial y} \right) + \mu \frac{\partial \lambda_4}{\partial y} \left(u \frac{\partial v'}{\partial x} + v \frac{\partial v'}{\partial y} \right) \\
& + \left(\tau_{xx} \frac{\partial \lambda_4}{\partial x} + \tau_{xy} \frac{\partial \lambda_4}{\partial y} \right) u' + \left(\tau_{xy} \frac{\partial \lambda_4}{\partial x} + \tau_{yy} \frac{\partial \lambda_4}{\partial y} \right) v' \\
& \left. + \gamma \frac{\mu}{\text{Pr}} \left(\frac{\partial e'}{\partial x} \frac{\partial \lambda_4}{\partial x} + \frac{\partial e'}{\partial y} \frac{\partial \lambda_4}{\partial y} \right) \right] d\Omega.
\end{aligned}$$

A compact form can be written as follows,

$$\begin{aligned}
\delta \mathcal{J}_4 = \int_{\Omega} & \left[(\vec{V} \cdot \vec{\nabla} \lambda_4) l(\vec{\nabla} \cdot \vec{V}') \right. \\
& + \mu u \vec{\nabla} \lambda_4 \cdot \vec{\nabla} u' + \mu v \vec{\nabla} \lambda_4 \cdot \vec{\nabla} v' + \mu \frac{\partial \lambda_4}{\partial x} \vec{V} \cdot \vec{\nabla} u' + \mu \frac{\partial \lambda_4}{\partial y} \vec{V} \cdot \vec{\nabla} v' \\
& + \left(\tau_{xx} \frac{\partial \lambda_4}{\partial x} + \tau_{xy} \frac{\partial \lambda_4}{\partial y} \right) u' + \left(\tau_{xy} \frac{\partial \lambda_4}{\partial x} + \tau_{yy} \frac{\partial \lambda_4}{\partial y} \right) v' \\
& \left. + \gamma \frac{\mu}{\text{Pr}} \left(\vec{\nabla} e' \cdot \vec{\nabla} \lambda_4 \right) \right] d\Omega.
\end{aligned}$$

The following vector identities are considered,

$$\begin{aligned}
(\vec{V} \cdot \vec{\nabla} \lambda_4) l(\vec{\nabla} \cdot \vec{V}') &= \vec{\nabla} \cdot (l(\vec{V} \cdot \vec{\nabla} \lambda_4) \vec{V}') - \vec{V}' \cdot \vec{\nabla} l(\vec{V} \cdot \vec{\nabla} \lambda_4), \\
\mu u \vec{\nabla} \lambda_4 \cdot \vec{\nabla} u' &= \vec{\nabla} \cdot \mu u u' \vec{\nabla} \lambda_4 - u' (\vec{\nabla} \cdot \mu u \vec{\nabla} \lambda_4), \\
\mu v \vec{\nabla} \lambda_4 \cdot \vec{\nabla} v' &= \vec{\nabla} \cdot \mu v v' \vec{\nabla} \lambda_4 - v' (\vec{\nabla} \cdot \mu v \vec{\nabla} \lambda_4), \\
\mu \frac{\partial \lambda_4}{\partial x} \vec{V} \cdot \vec{\nabla} u' &= \vec{\nabla} \cdot \mu \frac{\partial \lambda_4}{\partial x} u' \vec{V} - u' \left(\vec{\nabla} \cdot \mu \frac{\partial \lambda_4}{\partial x} \vec{V} \right), \\
\mu \frac{\partial \lambda_4}{\partial y} \vec{V} \cdot \vec{\nabla} v' &= \vec{\nabla} \cdot \mu \frac{\partial \lambda_4}{\partial y} v' \vec{V} - v' \left(\vec{\nabla} \cdot \mu \frac{\partial \lambda_4}{\partial y} \vec{V} \right), \\
\gamma \frac{\mu}{\text{Pr}} (\vec{\nabla} e' \cdot \vec{\nabla} \lambda_4) &= \frac{\gamma}{\text{Pr}} (\vec{\nabla} \cdot \mu e' \vec{\nabla} \lambda_4 - e' (\vec{\nabla} \cdot \mu \vec{\nabla} \lambda_4)),
\end{aligned}$$

for obtaining

$$\begin{aligned}
\delta\mathcal{J}_4 = & \int_{\Omega} \left[\vec{\nabla} \cdot (l(\vec{V} \cdot \vec{\nabla}\lambda_4)\vec{V}') + \vec{\nabla} \cdot \mu u u' \vec{\nabla}\lambda_4 + \vec{\nabla} \cdot \mu v v' \vec{\nabla}\lambda_4 \right. \\
& \left. + \vec{\nabla} \cdot \mu \frac{\partial\lambda_4}{\partial x} u' \vec{V} + \vec{\nabla} \cdot \mu \frac{\partial\lambda_4}{\partial y} v' \vec{V} + \frac{\gamma}{\text{Pr}} \vec{\nabla} \cdot \mu e' \vec{\nabla}\lambda_4 \right] d\Omega \\
& + \int_{\Omega} \left[\left(\tau_{xx} \frac{\partial\lambda_4}{\partial x} + \tau_{xy} \frac{\partial\lambda_4}{\partial y} - \frac{\partial}{\partial x} (l\vec{V} \cdot \vec{\nabla}\lambda_4) - \vec{\nabla} \cdot \mu u \vec{\nabla}\lambda_4 - \vec{\nabla} \cdot \mu \frac{\partial\lambda_4}{\partial x} \vec{V} \right) u' \right. \\
& \left. + \left(\tau_{xy} \frac{\partial\lambda_4}{\partial x} + \tau_{yy} \frac{\partial\lambda_4}{\partial y} - \frac{\partial}{\partial y} (l\vec{V} \cdot \vec{\nabla}\lambda_4) - \vec{\nabla} \cdot \mu v \vec{\nabla}\lambda_4 - \vec{\nabla} \cdot \mu \frac{\partial\lambda_4}{\partial y} \vec{V} \right) v' \right. \\
& \left. - \frac{\gamma}{\text{Pr}} (\vec{\nabla} \cdot \mu \vec{\nabla}\lambda_4) e' \right] d\Omega.
\end{aligned}$$

Applying the Gauss theorem and the no-slip boundary condition for the first integral, and introducing Ψ_{xx} , Ψ_{xy} , and Ψ_{yy} as

$$(A.22) \quad \Psi_{xx} = (l + 2\mu)u \frac{\partial\lambda_4}{\partial x} + lv \frac{\partial\lambda_4}{\partial y},$$

$$(A.23) \quad \Psi_{yy} = (l + 2\mu)v \frac{\partial\lambda_4}{\partial y} + lu \frac{\partial\lambda_4}{\partial x},$$

$$(A.24) \quad \Psi_{xy} = \mu \left(u \frac{\partial\lambda_4}{\partial y} + v \frac{\partial\lambda_4}{\partial x} \right),$$

gives

$$\begin{aligned}
(A.25) \quad \delta\mathcal{J}_4 = & - \int_{S_a} \gamma \frac{\mu}{\text{Pr}} (\vec{\nabla}\lambda_4 \cdot \vec{n}) e' dS \\
& + \int_{\Omega} \left[\left(\tau_{xx} \frac{\partial\lambda_4}{\partial x} + \tau_{xy} \frac{\partial\lambda_4}{\partial y} - \frac{\partial\Psi_{xx}}{\partial x} - \frac{\partial\Psi_{xy}}{\partial y} \right) u' \right. \\
& \left. + \left(\tau_{xy} \frac{\partial\lambda_4}{\partial x} + \tau_{yy} \frac{\partial\lambda_4}{\partial y} - \frac{\partial\Psi_{xy}}{\partial x} - \frac{\partial\Psi_{yy}}{\partial y} \right) v' \right. \\
& \left. - \frac{\gamma}{\text{Pr}} (\vec{\nabla} \cdot \mu \vec{\nabla}\lambda_4) e' \right] d\Omega.
\end{aligned}$$

Substituting equation (A.15) into e' in the surface integral and equation (A.14) into e' in the domain integral results in

$$\begin{aligned}
(A.26) \quad \delta\mathcal{J}_4 = & - \int_{S_a} \left[a^2 \frac{\mu}{\text{Pr}} (\vec{\nabla}\lambda_4 \cdot \vec{n}) \left(\frac{(\rho E)'}{p} - \frac{\rho'}{(\gamma-1)\rho} \right) \right] dS \\
& + \int_{\Omega} \left[\left(\tau_{xx} \frac{\partial\lambda_4}{\partial x} + \tau_{xy} \frac{\partial\lambda_4}{\partial y} - \frac{\partial\Psi_{xx}}{\partial x} - \frac{\partial\Psi_{xy}}{\partial y} \right) u' \right. \\
& \left. + \left(\tau_{xy} \frac{\partial\lambda_4}{\partial x} + \tau_{yy} \frac{\partial\lambda_4}{\partial y} - \frac{\partial\Psi_{xy}}{\partial x} - \frac{\partial\Psi_{yy}}{\partial y} \right) v' \right. \\
& \left. - \left(\frac{a^2 (\vec{\nabla} \cdot \mu \vec{\nabla}\lambda_4)}{(\gamma-1)\text{Pr}} \right) \left(\frac{p'}{p} - \frac{\rho'}{\rho} \right) \right] d\Omega.
\end{aligned}$$

Substitution of equations (A.10), (A.16), (A.21), and (A.26) into (A.6) yields

$$\begin{aligned}
(A.27) \quad \delta\mathcal{J} = & - \int_{S_a} \left[(\vec{\lambda} \cdot \vec{s})\tau'_w + (\vec{\lambda} \cdot \vec{n})\tau'_n \right. \\
& - \left(\mu \frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial s} - (\vec{\lambda} \cdot \vec{n})lH \right) (\vec{V}' \cdot \vec{n}) - \left(l \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} + (\vec{\lambda} \cdot \vec{s})\mu H \right) (\vec{V}' \cdot \vec{s}) \\
& + \lambda_4 \tau_w (\vec{V}' \cdot \vec{s}) + \lambda_4 a^2 \frac{\mu}{\text{Pr}} \left(\frac{\vec{\nabla}(\rho E)' \cdot \vec{n}}{p} - \frac{\vec{\nabla}\rho' \cdot \vec{n}}{(\gamma-1)\rho} \right) \\
& - \left(l(\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial n} \right) (\vec{V}' \cdot \vec{n}) - \mu \left(\frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} - H(\vec{\lambda} \cdot \vec{s}) \right) (\vec{V}' \cdot \vec{s}) \\
& \left. - a^2 \frac{\mu}{\text{Pr}} (\vec{\nabla} \lambda_4 \cdot \vec{n}) \left(\frac{(\rho E)'}{p} - \frac{\rho'}{(\gamma-1)\rho} \right) \right] dS \\
& + \int_{\Omega} \left[\left(\frac{\partial\Gamma_{xx}}{\partial x} + \frac{\partial\Gamma_{xy}}{\partial y} - \tau_{xx} \frac{\partial\lambda_4}{\partial x} - \tau_{xy} \frac{\partial\lambda_4}{\partial y} + \frac{\partial\Psi_{xx}}{\partial x} + \frac{\partial\Psi_{xy}}{\partial y} \right) u' \right. \\
& + \left(\frac{\partial\Gamma_{xy}}{\partial x} + \frac{\partial\Gamma_{yy}}{\partial y} - \tau_{xy} \frac{\partial\lambda_4}{\partial x} - \tau_{yy} \frac{\partial\lambda_4}{\partial y} + \frac{\partial\Psi_{xy}}{\partial x} + \frac{\partial\Psi_{yy}}{\partial y} \right) v' \\
& \left. + \left(\frac{a^2(\vec{\nabla} \cdot \mu \vec{\nabla} \lambda_4)}{(\gamma-1)\text{Pr}} \right) \left(\frac{p'}{p} - \frac{\rho'}{\rho} \right) \right] d\Omega.
\end{aligned}$$

The domain integral can be expressed in terms of the conservative flow variables by using the transformation

$$\mathbf{U}' = Y \mathbf{Q}',$$

where Y is the Jacobian of the primitive flow variables, $\mathbf{U} = (\rho \ u \ v \ p)^\top$, with respect to \mathbf{Q} ,

$$(A.28) \quad Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ \frac{-v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{(\gamma-1)(u^2+v^2)}{2} & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \end{bmatrix}.$$

The coefficients of \mathbf{U}' in equation (A.27) can be collected into a vector \mathbf{K} defined as

$$(A.29) \quad \mathbf{K} = \begin{pmatrix} -\frac{a^2(\vec{\nabla} \cdot \mu \vec{\nabla} \lambda_4)}{(\gamma-1)\text{Pr}} \frac{1}{\rho} \\ \frac{\partial\Gamma_{xx}}{\partial x} + \frac{\partial\Gamma_{xy}}{\partial y} - \tau_{xx} \frac{\partial\lambda_4}{\partial x} - \tau_{xy} \frac{\partial\lambda_4}{\partial y} + \frac{\partial\Psi_{xx}}{\partial x} + \frac{\partial\Psi_{xy}}{\partial y} \\ \frac{\partial\Gamma_{xy}}{\partial x} + \frac{\partial\Gamma_{yy}}{\partial y} - \tau_{xy} \frac{\partial\lambda_4}{\partial x} - \tau_{yy} \frac{\partial\lambda_4}{\partial y} + \frac{\partial\Psi_{xy}}{\partial x} + \frac{\partial\Psi_{yy}}{\partial y} \\ \frac{a^2(\vec{\nabla} \cdot \mu \vec{\nabla} \lambda_4)}{(\gamma-1)\text{Pr}} \frac{1}{p} \end{pmatrix}.$$

Finally, equation (A.27) is written as

$$\begin{aligned}
\text{(A.30)} \quad \delta\mathcal{J} = & - \int_{S_a} \left[(\vec{\lambda} \cdot \vec{n})\tau'_n + (\vec{\lambda} \cdot \vec{s})\tau'_w \right. \\
& - \left(\mu \frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial s} - (\vec{\lambda} \cdot \vec{n})lH \right) (\vec{V}' \cdot \vec{n}) - \left(l \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} + (\vec{\lambda} \cdot \vec{s})\mu H \right) (\vec{V}' \cdot \vec{s}) \\
& + \lambda_4 \tau_w (\vec{V}' \cdot \vec{s}) + \lambda_4 a^2 \frac{\mu}{\text{Pr}} \left(\frac{\vec{\nabla}(\rho E)' \cdot \vec{n}}{p} - \frac{\vec{\nabla} \rho' \cdot \vec{n}}{(\gamma - 1)\rho} \right) \\
& - \left(l(\vec{\nabla} \cdot \vec{\lambda}) + 2\mu \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial n} \right) (\vec{V}' \cdot \vec{n}) - \mu \left(\frac{\partial(\vec{\lambda} \cdot \vec{s})}{\partial n} + \frac{\partial(\vec{\lambda} \cdot \vec{n})}{\partial s} - H(\vec{\lambda} \cdot \vec{s}) \right) (\vec{V}' \cdot \vec{s}) \\
& \left. - a^2 \frac{\mu}{\text{Pr}} (\vec{\nabla} \lambda_4 \cdot \vec{n}) \left(\frac{(\rho E)'}{p} - \frac{\rho'}{(\gamma - 1)\rho} \right) \right] dS \\
& + \int_{\Omega} Y^\top \mathbf{K} \cdot \mathbf{Q}' d\Omega.
\end{aligned}$$

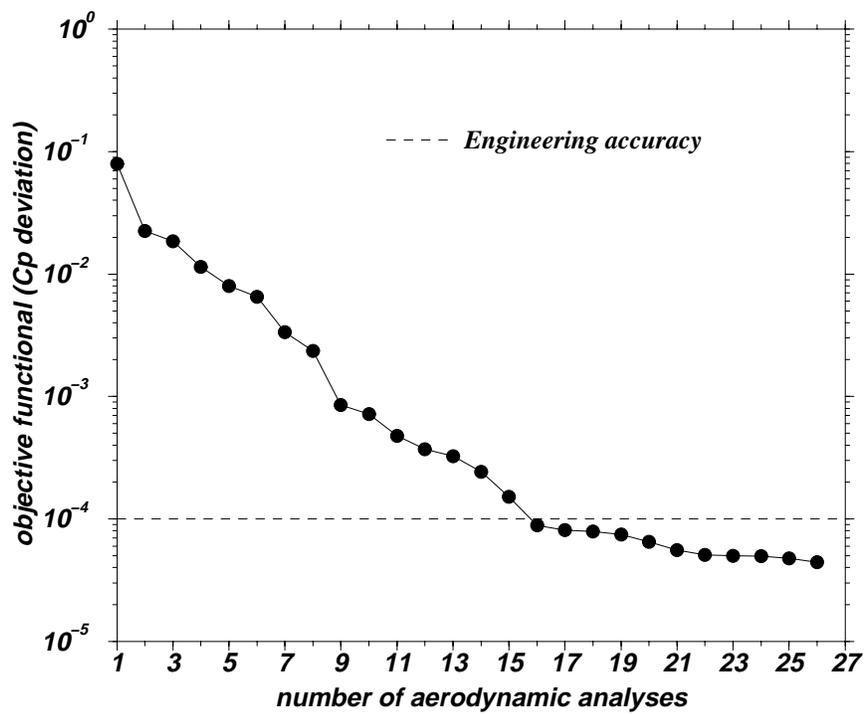


FIG. 5.1. Optimization history. $M = 0.73$, $\alpha = 2^\circ$, $Re = 6.5 \times 10^6$.

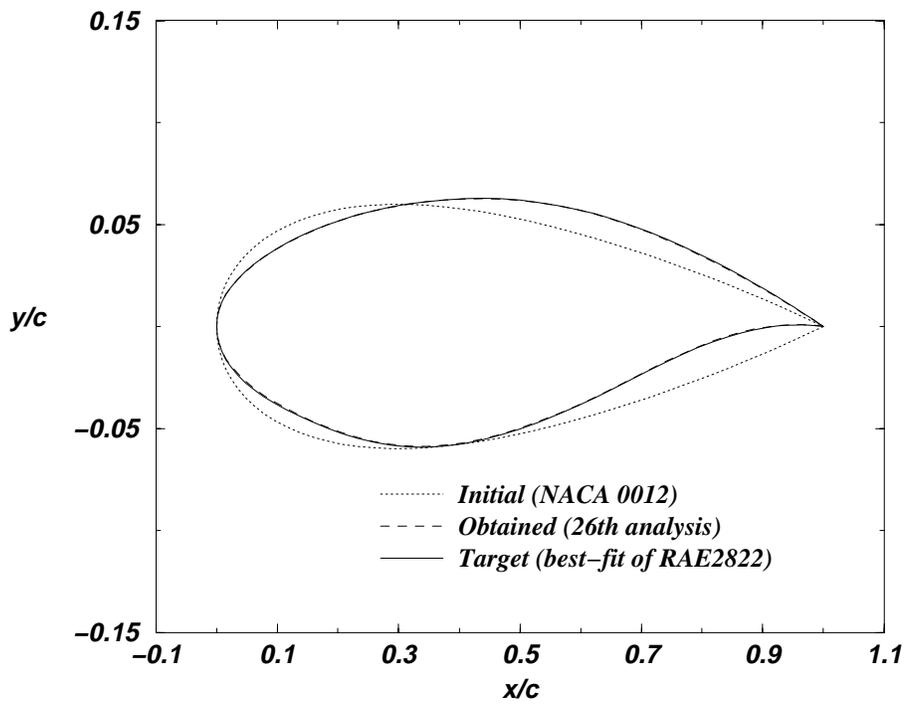
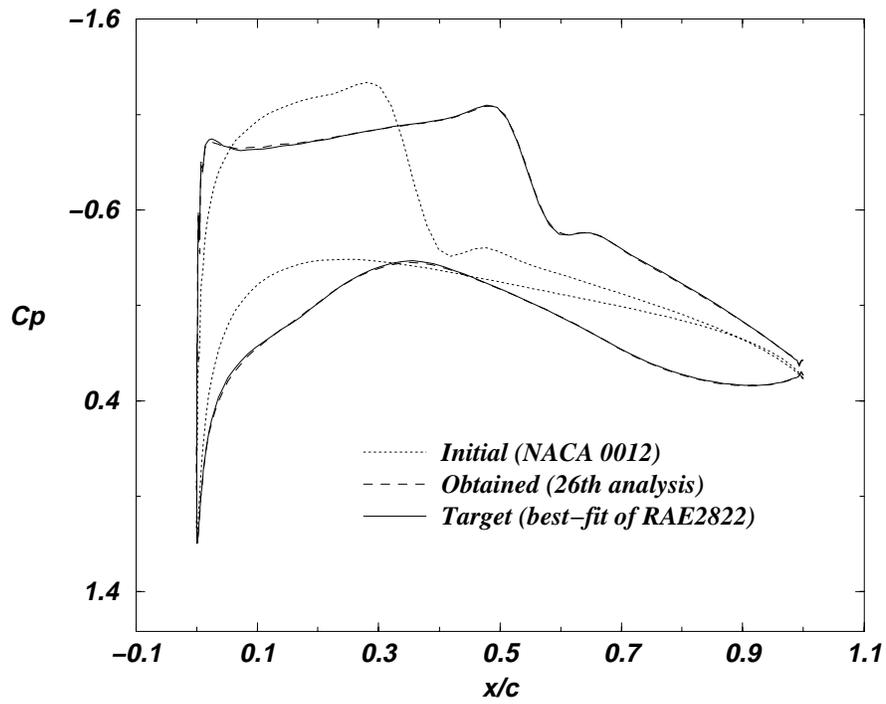


FIG. 5.2. C_p distribution and airfoil geometry. $M = 0.73$, $\alpha = 2^\circ$, $\text{Re} = 6.5 \times 10^6$.