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EFFECTS OF HELICITY ON LAGRANGIAN AND EULERIAN TIME CORRELATIONS IN TURBULENCE *

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Abstract. Taylor series expansions of turbulent time correlation functions are applied to show that helicity influences Eulerian time correlations more strongly than Lagrangian time correlations: to second order in time, the helicity effect on Lagrangian time correlations vanishes, but the helicity effect on Eulerian time correlations is nonzero. Fourier analysis shows that the helicity effect on Eulerian time correlations is confined to the largest inertial range scales. Some implications for sound radiation by swirling flows are discussed.

Key words. helicity, time correlations

Subject classification. Fluid Mechanics

1. Introduction. The theory of turbulent time correlations might be said to have begun with Kraichnan's demonstration [1] that the DIA response equation is infrared divergent when evaluated on a Kolmogorov inertial range. This calculation demonstrated the nonlocality of Eulerian time correlations. In later Lagrangian modifications [2] of DIA, the corresponding integrals converge; this confirmed the intuitive expectation [3] that Lagrangian time correlations are local in scale.

A simplification of Kraichnan's calculations was proposed by Kaneda [4] who observed that short time Taylor series expansions using the Navier-Stokes equations could produce useful information without explicitly invoking a specific closure scheme. These expansions express two-time correlations in terms of single-time correlations. Later, Kaneda and Gotoh [5] posed the question of the effect of helicity on Lagrangian and Eulerian time correlations. The present report proposes to complete this calculation using Kaneda's method and to note some possible applications.

2. Helical Turbulence. In the classic account of the kinematics of isotropic turbulence, Batchelor [6] noted the possibility that the turbulent random velocity field can lack mirror symmetry, and called this condition '*skew isotropy*.' Subsequent work used the term '*helicity*' instead. In terms of the single-time velocity correlation $Q_{ij}(\mathbf{k})$ defined by

$$< v_i(\mathbf{k},t) v_j(\mathbf{k}',t) >= Q_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}') \tag{1}$$

a helical random field is such that $Q_{ij}(\mathbf{k}) \neq Q_{ij}(-\mathbf{k})$, although necessarily $Q_{ij}(\mathbf{k}) = Q_{ji}(-\mathbf{k})$. Therefore, the correlation tensor is not symmetric: $Q_{ij}(\mathbf{k}) \neq Q_{ji}(\mathbf{k})$.

We will assume time-stationary, homogeneous, isotropic turbulence. For helical turbulence, the single-time correlation function is

$$Q_{ij}(\mathbf{k}) = Q(k)P_{ij}(\mathbf{k}) + i\epsilon_{imj}k_m H(k) \tag{2}$$

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Note that the *helicity spectrum* $H(k)$ contributes neither to the total energy of the velocity field nor more generally to any of the Reynolds stress components. A nonvanishing helicity spectrum is therefore consistent with the symmetry of the Reynolds stress tensor.

Define total helicity H by

$$H = < \mathbf{u} \cdot \nabla \times \mathbf{u} > = \int d\mathbf{k} \; 2k^2 H(k) \tag{3}$$

An elementary calculation [7] shows that total helicity is an inviscid invariant of the equations of motion. Helicity has a geometric significance in terms of the linking of vortex lines: if the vortex lines never link, the total helicity vanishes, but nonzero helicity indicates nontrivial linking of vortex lines. [7]

Although initial speculation that helicity plays a dynamic role in three-dimensional turbulence analogous to enstrophy in two-dimensional turbulence has not proven fruitful, interest in the possibility that helicity impedes energy transfer remains. Moffat and Tsinober [7] provide a recent review of helicity effects in turbulence.

3. Analysis. First note that

$$< u_p \frac{\partial u_i}{\partial x_q} > = \frac{1}{6} H \epsilon_{piq} \tag{4}$$

Let us begin by calculating the effect of helicity on Lagrangian single-point two-time correlations.[1] Let

$$u_i'(t) = \frac{Du_i}{Dt} \tag{5}$$

denote the convective derivative, so that

$$u_i'(t) = -\frac{\partial p}{\partial x_i} \tag{6}$$

Then the Lagrangian two-time correlation can be expanded in the Taylor series

$$< u_i(t) u_i(0) >_L = < u_i u_i > + < u_i' u_i > t + < u_i'' u_i > \frac{1}{2} t^2 + \cdots \tag{7}$$

where in Eq. (7) and subsequently, a velocity without a time argument indicates evaluation at time $t = 0$. In homogeneous turbulence, the linear term vanishes in view of Eq. (6), and stationarity in time implies

$$< u_i'' u_i > = - < u_i' u_i' > = - < \frac{\partial p}{\partial x_i} \frac{\partial p}{\partial x_i} > \tag{8}$$

It is convenient to write the last correlation as the Fourier transform

$$\begin{aligned} < \frac{\partial p}{\partial x_i} \frac{\partial p}{\partial x_i} > &= \int d\mathbf{k} \; k^{-2} \int \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) \delta(-\mathbf{k} - \mathbf{p}' - \mathbf{q}') d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \times \\ & < p_j q_i p_n' q_m' u_i(\mathbf{p}) u_j(\mathbf{q}) u_m(\mathbf{p}') u_n(\mathbf{q}') > \\ &= \int d\mathbf{k} \; k^{-2} k_j k_i k_m k_n \int \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) \delta(-\mathbf{k} - \mathbf{p}' - \mathbf{q}') d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \times \\ & < u_i(\mathbf{p}) u_j(\mathbf{q}) u_m(\mathbf{p}') u_n(\mathbf{q}') > \end{aligned} \tag{9}$$

Assuming the quasi-normal closure for fourth-order single-time correlations,

$$< \frac{\partial p}{\partial x_i} \frac{\partial p}{\partial x_i} > = 2 \int d\mathbf{k} \; k_i k_j k_m k_n k^{-2} \int \; \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) d\mathbf{p} d\mathbf{q} \; Q_{im}(\mathbf{p}) Q_{jn}(\mathbf{q}) \tag{10}$$

[1] We are grateful to Prof. Y. Kaneda for this argument.

On substituting Eq. (2) in Eq. (10), the symmetry of the wavevector factors in Eq. (10) implies that helicity makes no contribution to the second order expansion of the Lagrangian time correlation.

In the corresponding calculation for Eulerian correlations, let

$$\dot{u}_i = \frac{\partial u_i}{\partial t} \tag{11}$$

The obvious analogs of Eqs. (7)-(8) apply. Then

$$< \dot{u}_i \dot{u}_i > = < u_p \frac{\partial u_i}{\partial x_p} u_q \frac{\partial u_i}{\partial x_q} > +2 < u_i \frac{\partial p}{\partial x_i} > + < \frac{\partial p}{\partial x_i} \frac{\partial p}{\partial x_i} > \tag{12}$$

The second correlation on the right side vanishes by homogeneity, and it has just been shown that there is no helicity contribution to the the last term. Expanding the first correlation by quasi-normality,

$$\begin{aligned} < u_p \frac{\partial u_i}{\partial x_p} u_q \frac{\partial u_i}{\partial x_q} > &= < u_p \frac{\partial u_i}{\partial x_p} >< u_q \frac{\partial u_i}{\partial x_q} > + < u_p u_q >< \frac{\partial u_i}{\partial x_q} \frac{\partial u_i}{\partial x_q} > \\ &+ < u_p \frac{\partial u_i}{\partial x_q} >< u_q \frac{\partial u_i}{\partial x_p} > \\ &= \frac{2}{3} k\varepsilon/\nu - \frac{1}{6} H^2 \end{aligned} \tag{13}$$

The first term in Eq. (13) is the non-helical contribution to the time correlation and the second term shows the effect of helicity. Note that helicity decreases the turbulent frequency, hence it increases the Eulerian time scale.

This calculation can be refined somewhat by evaluating the same effect in wavevector space. Substituting Kolmogorov forms in Eq. (2),

$$Q_{ij}(\mathbf{k}) = C_K \varepsilon^{2/3} k^{-5/3} P_{ij}(\mathbf{k})/4\pi k^2 + C_H \varepsilon_H^{2/3} i\epsilon_{imj} k_m k^{-7/3}/4\pi k^3 \tag{14}$$

Then

$$H = 6 C_H \varepsilon_H^{2/3} k_0^{-1/3} \tag{15}$$

where $2\pi k_0^{-1}$ is the integral scale of the turbulence.

The quadratic term in Kaneda's expansion is

$$< \ddot{u}_i(\mathbf{k}) u_j(-\mathbf{k}) > = -\frac{1}{4} \int \ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \ P_{imn}(\mathbf{k}) P_{mrs}(\mathbf{p}) Q_{rj}(\mathbf{k}) Q_{ns}(\mathbf{q}) \tag{16}$$

We will investigate how this integral diverges in the limit $\mathbf{q} \to 0$, isolating the helical contribution. In this limit, the contribution due to helicity to the integrand in Eq. (16) is

$$\begin{aligned} P_{imn}(\mathbf{k}) P_{mrs}(\mathbf{p}) Q_{rj}(\mathbf{k}) Q_{ns}(\mathbf{q}) &= -P_{imn}(\mathbf{k})\{P_{mrs}(\mathbf{k}) - q_r P_{ms}(\mathbf{k}) \\ - q_s P_{mr}(\mathbf{k}) + k_r q_a P_{ams}(\mathbf{k}) k^{-2} &+ k_s q_a P_{amr}(\mathbf{k}) k^{-2}\} \times \\ &\epsilon_{rj\alpha} \epsilon_{ns\beta} k_\alpha q_\beta \varepsilon_H^{4/3} (kq)^{-10/3}/16\pi^2 k^2 q^2 \end{aligned} \tag{17}$$

Then a simple calculation shows that

$$\begin{aligned} &-P_{imn}(\mathbf{k})\{P_{mrs}(\mathbf{k}) - q_r P_{ms}(\mathbf{k}) - q_s P_{mr}(\mathbf{k}) \\ &+ k_r q_a P_{ams}(\mathbf{k}) k^{-2} + k_s q_a P_{amr}(\mathbf{k}) k^{-2}\} \epsilon_{rj\alpha} \epsilon_{ns\beta} k_\alpha q_\beta \\ &= q_{r\beta} k_{n\alpha} \epsilon_{rj\alpha} \epsilon_{ni\beta} - k_{s\alpha} q_{r\beta} \epsilon_{rj\alpha} \epsilon_{is\beta} \\ &= -2 o_i o_j \end{aligned} \tag{18}$$

where

$$o_i = \epsilon_{ipq} k_p q_q \tag{19}$$

The integral with respect to $\mathbf{q}$ in Eq. (16) therefore diverges as

$$\int dq\ q^{-10/3+2} \sim k_0^{-1/3} \tag{20}$$

consequently, for the helical contribution to the second order term,

$$< \ddot{u}_i(\mathbf{k}) u_j(-\mathbf{k}) > \sim P_{ij}(\mathbf{k}) k^2 \varepsilon_H^{2/3} k^{-10/3} \varepsilon_H^{2/3} k_0^{-1/3} = P_{ij}(\mathbf{k}) E_H(k) H \tag{21}$$

Integration of this result over k recovers the H^2 dependence in Eq. (13) of the helical contribution to the single-point correlation. Note the non-local dependence on the total helicity, which is analogous to the non-local dependence of Eulerian time correlations on the total rms velocity fluctuations.

Define the Eulerian frequency scale by

$$\omega_E^2 = \ - < \ddot{u}_i(\mathbf{k}) u_i(-\mathbf{k}) > /Q(k) \tag{22}$$

Then adding the usual sweeping contribution gives the complete Eulerian frequency scale to second order

$$\omega_E = [C_D^2 (Vk)^2 - C'_H H \frac{\varepsilon_H^{2/3}}{\varepsilon^{2/3} k^{1/3}}]^{1/2} = C_D V k \{1 - \frac{C'_H}{2C_D^2} \frac{H(\varepsilon_H/\varepsilon)^{2/3}}{V^2} k^{-5/3}\} \tag{23}$$

Again, the frequency is decreased by helicity. Note that the effect of helicity is confined to the largest inertial range scales.

4. Conclusion. This calculation reveals an additional dynamic distinction between Eulerian and Lagrangian time correlations. The agreement to second order of the Taylor series expansion of helical and non-helical Lagrangian time correlation functions suggests that the effect of helicity on Lagrangian time correlations is relatively weak, whereas helicity strongly alters Eulerian correlations.

The dependence of turbulent energy transfer on Lagrangian correlations [2, 3] might seem to suggest a relatively small effect of helicity on turbulent energy transfer in steady-state flows. However, closures suggest that both spatial and temporal correlations influence energy transfer, so that this analysis is certainly consistent with some connection between helicity and energy transfer. These connections are discussed in [7, 8]. It is possible that helicity could stabilize turbulence and play a role in the sustainment of large-scale structures, as suggested by Yokoi and Yoshizawa. [9]

Although less theoretical interest is naturally attached to Eulerian time correlations, they arise in problems including wave propagation through turbulence sound radiation by turbulence. In the sound radiation problem, the Eulerian space-time correlation function appears in the turbulent sound source in Lighthill's theory. [10] The effect of helicity on radiated sound is a redistribution of acoustic energy radiated by the largest inertial range scales to lower frequencies. In helical flows, like swirling jets, helicity may also greatly alter the large structures in the flow [9] and consequently modify the sweeping velocity. The combination of both helicity effects may therefore cause significant modification of sound radiation from swirling flows.

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