NASA/CR-2000-210124
ICASE Report No. 2000-25

Trust-region Proper Orthogonal Decomposition for Flow Control

E. Arian
Ohio University, Athens, Ohio

M. Fahl and E.W. Sachs
Universität Trier, Trier, Germany

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA

Operated by Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

Prepared for Langley Research Center
under Contract NAS1-97046

May 2000

TRUST-REGION PROPER ORTHOGONAL DECOMPOSITION FOR FLOW CONTROL

E. ARIAN*, M. FAHL†, AND E.W. SACHS‡

Abstract. The proper orthogonal decomposition (POD) is a model reduction technique for the simulation of physical processes governed by partial differential equations, e.g. fluid flows. It can also be used to develop reduced order control models. Fundamental is the computation of POD basis functions that represent the influence of the control action on the system in order to get a suitable control model. We present an approach where suitable reduced order models are derived successively and give global convergence results.

Key words. proper orthogonal decomposition, Navier-Stokes, flow control, reduced order modeling, trust region

Subject classification. Applied Numerical Mathematics

1. Introduction. We present a robust reduced order method for the control of complex time-dependent physical processes governed by partial differential equations (PDE). Such a control problem often is hard to solve because of the high order system that describes the state (a large number of (finite element) basis elements for every point in the time discretization). The proper orthogonal decomposition (POD) is a reduced order modeling approach that has been successfully applied for the simulation and control of complex systems, see e.g. [1, 3, 4, 5, 6, 10, 11, 13]. POD based reduced order models are used to avoid the difficulty of dealing with large systems by using global basis functions instead of local basis functions for the Galerkin projection of the considered PDE. Often a small number of these global basis functions suffices to obtain a satisfactory level of accuracy.

However, the limited number of degrees of freedom in the reduced order POD model constitutes its main weakness for optimal control purposes. Since the POD model is based on the solution of the PDE for a specified control, it might be a poor model when the controller takes the system from its original state towards the optimal state. There is no guarantee that the reduced order process will converge to the optimal control of the original (large) system. This difficulty leads us to propose a Trust-Region Proper Orthogonal Decomposition (TRPOD) method, that constructs successively improved POD models based on the updated control values. By embedding the POD technique into the concept of trust-region (TR) methods with general, non-quadratic, model functions and inexact gradient information [8, 16] we are able to prove convergence of the proposed scheme. That approach is related with optimization methods that use surrogate objectives [2, 7].

We concentrate on the application of TRPOD for flow control problems with Dirichlet boundary control governed by the time-dependent Navier-Stokes equations (NSE) for viscous incompressible fluids.

*Ohio University, Department of Mathematics, Athens, Ohio, (arian@math.ohiou.edu). This research was supported in part by the National Aeronautics and Space Administration under NASA Contract No. NAS1-97046 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, Virginia 23681.

†Universität Trier, FB IV - Mathematik, Trier, Germany, (fahl@uni-trier.de). This research was supported by the "Deutsche Forschungsgemeinschaft" through the Graduiertenkolleg "Mathematische Optimierung" at the University of Trier, and in part by the National Aeronautics and Space Administration under NASA Contract No. NAS1-97046 while the author was a visitor at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, Virginia 23681.

‡Universität Trier, Fachbereich IV, Abteilung Mathematik, 54286 Trier, Germany (sachs@uni-trier.de).

The paper is organized as follows. In section 2 we review the basic ideas of the POD and derive a POD based control model. In section 3 the ideas of TRPOD modeling are presented. In section 4 global convergence of the TRPOD scheme is treated. In section 5 several numerical examples are given. In section 6 we give a short discussion and some concluding remarks.

2. Reduced Order Modeling Using Proper Orthogonal Decomposition (POD). The starting point for POD based reduced order modeling, for flow problems, is the availability of an input collection of flow fields $y^i(x) = y(x, t_i)$ (snapshots). Given the snapshot set

$$\mathcal{Y}^{SNAP} = \{y^1, \ldots, y^N\}, \tag{2.1}$$

the POD technique can be used to identify dominant spatial structures in the flow. For ease of notation we assume that the snapshots are linearly independent. Then the POD procedure computes an orthonormal basis

$$\mathcal{Y}^{POD} = \{\psi^1, \ldots, \psi^N\}$$

for the subspace spanned by $\mathcal{Y}^{SNAP}$ and determines the M most important basis elements for the representation of the snapshot set in the above orthonormal basis.

The (accumulated) error of this optimal truncated basis representation is given by

$$\varepsilon^{POD}(M) = \sum_{i=1}^{N} ||y^i - \sum_{j=1}^{M} (y^i, \psi^j)\psi^j||^2 = \sum_{i=1}^{N} || \sum_{j=M+1}^{N} (y^i, \psi^j)\psi^j||^2 \tag{2.2}$$

where

$$y^i = \sum_{j=1}^{N} (y^i, \psi^j)\psi^j, \quad i \in \{1, \ldots, N\},$$

and can be computed explicitly using singular value analysis [6, 10, 14]. By the choice of M a control of the magnitude of $\varepsilon^{POD}(M)$ is possible. We call $\mathbf{\Psi} = \{\psi^1, \ldots, \psi^M\}$ a POD basis of order M.

Expanding the velocity field in terms of the POD basis,

$$y(x, t) = \sum_{j=1}^{M} a_j(t)\psi^j(x),$$

and projecting the Navier-Stokes equations onto the subspace spanned by the POD basis yields an M-dimensional ODE system for the expansion coefficients:

$$\dot{a}(t) = F(a(t), t), \quad a(0) = a^0. \tag{2.3}$$

We call (2.3) a POD model of order M. Combining the solution of (2.3) and the expansion (2.2) results in an approximation of the flow dynamics.

Let us suppose that the snapshot set (2.1) corresponds to a flow behavior forced by a certain Dirichlet boundary control value, so we can replace (2.3) with the following (cf. [9, 13, 15])

$$\dot{a}(t) = F(a(t), u(t), t), \quad a(0) = a^0. \tag{2.4}$$

Here, we assumed the control to be of the following form (u denotes the Dirichlet data and g is a fixed function):

$$\mathrm{u}(x, t) = u(t)g(x) \quad \text{on } \Gamma_c.$$

We call (2.4) a POD based control model of order M.

3. The Trust-Region Proper Orthogonal Decomposition (TRPOD) Optimal Control Approach. In the following we consider the problem of minimizing

$$f(\mathtt{u}) = \frac{1}{2}\int_0^T \int_{\Omega_{obs}} |y(\mathtt{u}) - y^d|^2 \, dx \, dt \tag{3.1}$$

where $y(x,t) = y(\mathtt{u}; x, t)$ is the solution to the two dimensional Navier-Stokes equations for prescribed Dirichlet boundary data, $\mathtt{u}(x,t)$, $y^d(x,t)$ is a desired state and $\Omega_{obs} \subset \Omega$ is the observation volume for the observation of the state on the domain Ω. With regard to the POD based reduced order modeling approach we would ideally like to substitute the 'full' state equations with a POD based control model that could be used for the optimal control process. In this case (3.1) is replaced with

$$f^{POD}(u) = \frac{1}{2}\int_0^T \int_{\Omega_{obs}} |y^{POD}(u) - y^d|^2 \, dx \, dt \tag{3.2}$$

where $y^{POD}(u; x, t)$ is the solution of the POD reduced order model for control value $u(t)$ (using (2.4) as the POD based control model).

A priori it is not clear what is the best way to generate snapshots that are useful for the POD based control procedure. A successful POD based control model should represent correctly the dynamics of the flow that is altered by the controller. It is therefore natural to improve the POD based control model successively by improving the snapshot set that is used to generate the POD basis.

First, we start with an 'arbitrary' initial control $u_0(t)$ and compute snapshots that correspond to the flow behavior forced by $u_0(t)$. We use these snapshots to compute a first POD basis $\mathbf{\Psi}^{(0)}$ and to build up a corresponding POD based control model. We denote the optimal control based on $\mathbf{\Psi}^{(0)}$ by $u_1(t)$. If we use $u_1(t)$ for the computation of a new snapshot set and a new POD basis $\mathbf{\Psi}^{(1)}$ we can improve the initial POD based control model. The new basis is an improvement in the sense that the minimization of the objective function based on $\mathbf{\Psi}^{(1)}$ results in an 'optimal' control $u_2(t)$ such that $f(u_2) \leq f(u_1)$. The solution process for the reduced order control problem with respect to this second POD based control model can be started with the last control function iterate $u_1(t)$.

If we proceed this way, i.e, given a current iterate, $u_k(t)$, for the control we compute corresponding snapshots, $\{y_k^1, \ldots, y_k^{N_k}\}$, compute a POD basis $\mathbf{\Psi}^{(k)}$, build a current POD based control model $(PCM)_k$

$$(PCM)_k \qquad \dot{a}(t) = F(a(t), u_k(t), t), \quad a(0) = a^0 \tag{3.3}$$

and use this model for the computation of the next iterate $u_{k+1}(t)$ via the minimization of (3.2), we expect the POD model to converge to a model that represents the flow at the optimal control of (3.1).

In order to guarantee the convergence of the above process we additionally embed it in a trust region framework.

For given control value u_k we add a trust region constraint to the unconstrained optimization problem (3.2) and minimize with respect to s

$$f^{POD}(u_k + s) = \frac{1}{2}\int_0^T \int_{\Omega_{obs}} |y^{POD}(u_k + s) - y^d|^2 \, dx \, dt \tag{3.4}$$

subject to

$$||s||_{L^2} \leq \delta_k. \tag{3.5}$$

Here, $y^{POD}(u_k + s; x, t)$ denotes the reduced order flow solution for given control value $u_k + s$, where the POD based control model is generated from the snapshots related to u_k. δ_k denotes the trust region radius at iteration k.

We define the local nonlinear trust-region model for the original objective f at point u_k by:

$$m_k(u_k + s) := f^{POD}(u_k + s). \tag{3.6}$$

We denote the gradient of the model function at the center point of the trust region by $g_k := \nabla m_k(u_k)$. g_k is an approximation to the gradient of the objective function f.

The outline of the resulting TRPOD algorithm is given below (a stopping criterion should be added in practice).

Algorithm 1: Outline of the TRPOD algorithm

Let u_0, δ_0, $0 < \eta_1 < \eta_2 < 1$ and $0 < \gamma_1 \leq \gamma_2 < 1 \leq \gamma_3$ be given, set $k = 0$.

1. Compute snapshot set $\mathcal{Y}_k^{SNAP}$ corresponding to control u_k.

2. Compute POD basis $\mathbf{\Psi}^{(k)}$ and build POD based control model (3.3).

3. Minimize the model function within the trust region

$$s_k = \arg \min_{||s|| \leq \delta_k} m_k(u_k + s)$$

4. Compute $f(u_k + s_k)$ and set and

$$\rho_k = \frac{f(u_k) - f(u_k + s_k)}{m_k(u_k) - m_k(u_k + s_k)}$$

5. Update the trust-region radius:
   - If $\rho_k \geq \eta_2$: set $u_{k+1} = u_k + s_k$ and increase trust region radius $\delta_{k+1} = \gamma_3 \delta_k$, set $k = k + 1$ and GOTO 1.
   - If $\eta_1 < \rho_k < \eta_2$: set $u_{k+1} = u_k + s_k$ and decrease trust region radius $\delta_{k+1} = \gamma_2 \delta_k$, set $k = k + 1$ and GOTO 1.
   - If $\rho_k \leq \eta_1$: set $u_{k+1} = u_k$ and decrease trust region radius $\delta_{k+1} = \gamma_1 \delta_k$, set $k = k + 1$ and GOTO 3.

Following [16] we call an iteration *successful* if $\rho_k > \eta_1$, i.e., the actual reduction

$$ared_k(s_k) = f(u_k) - f(u_k + s_k)$$

is large enough compared to the predicted reduction

$$pred_k(s_k) = m_k(u_k) - m_k(u_k + s_k).$$

In the case $\rho_k \leq \eta_1$, we call the iteration *unsuccessful*.

For a practical implementation of the above algorithm, in the spirit of trust region methods, two specific modifications should be considered.

First, in the case of a rejection of the new trial step we can adapt the reduction of the trust region radius to the step length of the unsuccessful step, in order to avoid a possible number of unnecessary reductions that show no effect in the computation of a new descent direction.

Second, it is not necessary to compute an exact (global) minimum in step 3 of Algorithm 1. Instead, we use the following step determination algorithm for the computation of a descent direction [16].

Algorithm 2: Step determination algorithm

Let

$$0 < \alpha < \beta < 1,\ 0 < \nu_1 < 1,\ \nu_2 > 0,\ \nu_3 > 0,\ 0 < \mu \leq 1 \tag{3.7}$$

be given.

Phase 1: Find λ_k^A such that

$$m_k(u_k - \lambda_k^A g_k) \leq m_k(u_k) - \alpha \lambda_k^A ||g_k||^2 \tag{3.8}$$

$$||\lambda_k^A g_k|| \leq \delta_k \tag{3.9}$$

and

$$\lambda_k^A \geq \lambda_k^B \ \text{ or } \ \lambda_k^A \geq \min\{\nu_1 \frac{\delta_k}{||g_k||}, \nu_2\} \tag{3.10}$$

where $\lambda_k^B > 0$ (if required) must satisfy

$$m_k(u_k - \lambda_k^B g_k) \geq m_k(u_k) - \beta \lambda_k^B ||g_k||^2 \tag{3.11}$$

Phase 2: If $\delta_k \geq \nu_3$: Choose step s_k such that

$$m_k(u_k) - m_k(u_k + s_k) \geq \mu \left(m_k(u_k) - m_k(u_k - \lambda_k^A g_k)\right) \tag{3.12}$$

$$||s_k|| \leq \delta_k \tag{3.13}$$

In Phase 1 of Algorithm 2 we compute a step in the steepest descent direction that guarantees a sufficient decrease in the model function. We also ensure that the step stays within the trust region and that the step size is not too small. This means that Phase 1 can be interpreted as a substitute for the computation of the Cauchy point in standard trust region methods for unconstrained optimization (UTR). Phase 2 allows to leave the steepest descent direction if the trust region is sufficiently large. Similar to dogleg methods for the approximation of $s_k = \arg\min_{||s|| \leq \delta_k} m_k(u_k + s)$ in the quadratic model function case by using a descent direction that includes a certain part of the step into the steepest descent direction and the (Quasi-) Newton direction, the above algorithm first guarantees a sufficient decrease in the model function by using the steepest descent direction (Phase 1) and then it allows an optional modification of the step direction in Phase 2. For Algorithm 2 the following result holds whose proof can be found in [16].

LEMMA 3.1 (Toint '88). *Provided that (3.7) holds, there always exists a step s_k satisfying the conditions (3.8) - (3.13).*

4. Convergence Results. In the context of trust region methods we can interpret the TRPOD approach for the optimal flow control problem as a trust region method with a non-quadratic model function [16].

For this purpose we consider a discretization of (3.1), (3.6) such that $f, m_k : \mathbb{R}^n \to \mathbb{R}$. For a given u_0 in the control space $\mathcal{U}$, let $\mathcal{U}_0$ be an open convex subset containing the level set $\mathcal{L}_0$ defined by:

$$\mathcal{L}_0 = \left\{ u \in \mathcal{U} \middle| f(u) \leq f(u_0) \right\} \subset \mathcal{U}_0 \subseteq \mathcal{U}.$$

We assume that f satisfies the following standard assumptions:

- f is continuously differentiable on $\mathcal{U}_0$,
- f is bounded below on $\mathcal{U}_0$,
- ∇f is Lipschitz continuous with constant L on $\mathcal{U}_0$.

We emphasize the fact that unlike common quadratic trust region model functions $m_k(u_k + s)$ is not quadratic in s. Furthermore, for the model function m_k based on POD we have

$$m_k(u_k) \neq f(u_k) \tag{4.1}$$

and

$$g_k = \nabla m_k(u_k) \neq \nabla f(u_k), \tag{4.2}$$

i.e., both the model function value at the trust region's center point and the gradient of the POD derived model function are inexact. We assume the models m_k to satisfy the following properties [8, 16]:

- each m_k is differentiable
- there exists a positive integer, N, such that the gradient of each model, g_k, satisfies the following inequality for $k > N$,

$$\frac{|(g_k - \nabla f(u_k))^T s_k|}{||g_k||\,||s_k||} \leq \zeta \tag{4.3}$$

  for some user-specified constant $\zeta > 0$, where s_k is the step according to Alg. 2.

Condition (4.3) requires that the normalized reduced order directional derivative in descent direction approximates the normalized full directional derivative sufficiently well in the limit.

Based on the above assumptions we prove the convergence of the TRPOD scheme using techniques similar to the standard trust region method for unconstrained optimization. Specifically, we use results of Toint [16] concerning trust region algorithms with non-quadratic model functions, and Carter [8] who also treated trust region algorithms with inexact gradient information.

For convergence proofs of standard trust region methods with inexact gradients an essential condition is the sufficient decrease condition

$$f(u_k) - f(u_k + s_k) \geq \frac{1}{2} c\, ||g_k|| \min\{\delta_k, \frac{||g_k||}{||H_k||}\} \tag{4.4}$$

for some $c > 0$ and where $||H_k||$ denotes the norm of the Hessian of the model function.

In the general nonlinear model function case we have to derive an analogous sufficient decrease condition by other means. We define the following estimate of a function's curvature along the step s based at a point u [16]:

$$\omega(f, u, s) := \frac{2}{||s||^2}(f(u + s) - f(u) - \nabla f(u)^T s). \tag{4.5}$$

The following theorem gives a lower bound to the predicted decrease in the model function [16].

THEOREM 4.1 (Toint '88). *Assume that* $||g_k|| \neq 0$ *and define according to Algorithm 2*

$$\omega_k := \begin{cases} 0 & \text{when } \lambda_k^B \text{ is undefined} \\ \omega(m_k, u_k, -\lambda_k^B g_k) & \text{when } \lambda_k^B \text{ is defined} \end{cases} . \tag{4.6}$$

Then

$$\omega_k \geq 0$$

and there exists a constant $c_s > 0$ *such that Algorithm 2 produces a step* s_k *with*

$$m_k(u_k) - m_k(u_k + s_k) \geq c_s ||g_k||^2 \min\{||g_k||^2/(1+\omega_k), \delta_k\}. \tag{4.7}$$

As an immediate corollary we get the following result, analogous to the sufficient decrease condition (4.4), that describes the decrease in the objective when the iteration is successful [16].

COROLLARY 4.2. *Let the assumptions of Theorem 4.1 be satisfied and assume that the* k*-th iteration was successful. Then*

$$f(u_k) - f(u_{k+1}) \geq \eta_1 c_s ||g_k||^2 \min\{||g_k||^2/(1+\omega_k), \delta_k\}.$$

The next lemma is required in the global convergence proof (cf. [8]). Its proof follows from the definition of Algorithm 2.

LEMMA 4.3. *Let* $\{s_k\}$ *be a sequence of steps computed by Algorithm 2 and* $\{g_k\}$ *the sequence of the model gradients. We define*

$$\cos\Theta_k := \frac{-s_k^T g_k}{||s_k||\,||g_k||}. \tag{4.8}$$

If $\liminf_{k\to\infty} ||g_k|| > 0$ *and* $\lim_{k\to\infty} \delta_k = 0$ *then*

$$\lim_{k\to\infty} \cos\Theta_k = 1. \tag{4.9}$$

We now give the main convergence theorem for the TRPOD method.

THEOREM 4.4. *Let* f *satisfy the standard assumptions. Assume that* $\{u_k\}$ *is a sequence of iterates generated by Algorithm 1 with step determination according to Algorithm 2 and that there exists a positive integer,* N*, such that*

$$\frac{|(g_k - \nabla f(u_k))^T s_k|}{||g_k||\,||s_k||} \leq \zeta \quad \text{for all } k > N \tag{4.10}$$

for some ζ *with* $0 < \zeta < 1 - \eta_2$*. We define*

$$b_k := 1 + \max\{\max\{\omega_i, |\omega(m_i, u_i, s_i)|\}, i = 0, \ldots, k\} \tag{4.11}$$

and assume that there exists some constant $c_b > 0$ *such that*

$$b_k \leq c_b \quad \text{for all } k > N. \tag{4.12}$$

Then

$$\liminf_{k\to\infty} ||g_k|| = 0.$$

Proof: The proof follows by contradiction. We assume that

$$\liminf_{k\to\infty} ||g_k|| \geq \varepsilon \tag{4.13}$$

for some $\varepsilon > 0$. Using Corollary 4.2 we get

$$\sum_{k=0}^{\infty} \delta_k < \infty \tag{4.14}$$

using standard arguments [8]. For k sufficiently large with (4.5) and (4.8) we have

$$\begin{aligned}
1-\rho_k &= \frac{m_k(u_k+s_k) - m_k(u_k) - (f(u_k+s_k) - f(u_k))}{m_k(u_k+s_k) - m_k(u_k)} \\
&= \frac{\frac{1}{2}||s_k||^2\omega(m_k,u_k,s_k) + g_k^T s_k - (\frac{1}{2}||s_k||^2\omega(f,u_k,s_k) + \nabla f(u_k)^T s_k)}{\frac{1}{2}||s_k||^2\omega(m_k,u_k,s_k) + g_k^T s_k} \\
&\leq \frac{|(\nabla f(u_k) - g_k)^T s_k| + \frac{1}{2}||s_k||^2|\omega(f,u_k,s_k) - \omega(m_k,u_k,s_k)|}{-g_k^T s_k - \frac{1}{2}||s_k||^2\omega(m_k,u_k,s_k)} \\
&\leq \frac{|(\nabla f(u_k) - g_k)^T s_k| + \frac{1}{2}||s_k||^2(|\omega(f,u_k,s_k)| + |\omega(m_k,u_k,s_k)|)}{||g_k||\,||s_k||\cos\Theta_k - \frac{1}{2}||s_k||^2\omega(m_k,u_k,s_k)} \\
&= \frac{\frac{|(\nabla f(u_k)-g_k)^T s_k|}{||g_k||\,||s_k||} + \frac{1}{2}\frac{||s_k||}{||g_k||}(|\omega(f,u_k,s_k)| + |\omega(m_k,u_k,s_k)|)}{\cos\Theta_k - \frac{1}{2}\frac{||s_k||}{||g_k||}\omega(m_k,u_k,s_k)} \qquad (4.15)
\end{aligned}$$

The Lipschitz continuity of ∇f leads to

$$\omega(f,u_k,s_k) \leq L \tag{4.16}$$

where L is the Lipschitz constant of ∇f. Combining (4.16), (4.11) and $||s_k|| \leq \delta_k$ yields

$$1-\rho_k \leq \frac{\frac{|(\nabla f(u_k)-g_k)^T s_k|}{||g_k||\,||s_k||} + \frac{1}{2}\frac{||s_k||}{||g_k||}(L+b_k)}{\cos\Theta_k - \frac{1}{2}\frac{||s_k||}{||g_k||}\omega(m_k,u_k,s_k)}. \tag{4.17}$$

Now we can use (4.9), (4.10),(4.11), (4.12), (4.13) and $\delta_k \to 0$ to deduce that

$$\lim_{k\to\infty}(1-\rho_k) \leq \zeta < 1-\eta_2. \tag{4.18}$$

The above limit implies that there exists $\bar{k}$ with $\rho_k > \eta_2$ for all $k \geq \bar{k}$ such that the trust region radius δ_k is not reduced for all $k \geq \bar{k}$, leading to a contradiction with $\delta_k \to 0$ for $k \to \infty$. □

Remark: Since $\omega(m_k,u_k,s_k)$ is a measure for the model's curvature in step direction, assumption (4.12) is an analogue to the standard assumption of uniformly boundedness of the models Hessian in the quadratic model function case (cf. [12]).

TABLE 5.1
TRPOD Parameters

TR Framework		Step Determination	
$\eta_1,\ \eta_2$	0.25, 0.75	$\alpha,\ \beta$	0.25, 0.75
$\gamma_1,\ \gamma_2,\ \gamma_3$	0.5, 0.5, 2	$\nu_1,\ \nu_2,\ \nu_3,\ \mu$	0.1, 0.1, 0.1 , 1

5. Numerical Results. In this section we present numerical studies of the TRPOD based control method applied to the standard driven cavity test case. All flow calculations were carried out with the flow solver FEATFLOW [17] at Reynolds number $Re = 200$, on the time interval $[0, 5]$, using a uniform grid. The parameters for the TRPOD algorithm are given in Table 5.1.

Two examples are in order. In the first example we choose a small observation domain (1.44% of Ω) and a small control space (7 control variables). In that example the problem is small enough so that the assumptions of Theorem 4.4 can be verified explicitely. In the second example we choose the entire cavity to be the observation domain and the control space is of maximum dimension (167 control variables). The purpose of this example to illustrate the significant change in the POD basis due to the changes in the control input, and to demonstrate the effectiveness of the TRPOD method for larger scale problems. In that example we did not verify the assumptions of Theorem 4.4 due to the large number of control variables.

Example 1. In the first example the control of the cavity flow is the top wall velocity (and there is no control action on the bottom wall, i.e., $y^1 = y^2 = 0$ on Γ_{bot}). We reduce the number of control variables by restricting the control space using the expansion $u(t) = \sum_{k=1}^{R} u_k \varphi_k(t)$ with $\varphi_1(t) \equiv 1$ and $\varphi_{2k}(t) = \cos(2k\pi t/T)$, $\varphi_{2k+1}(t) = \sin(2k\pi t/T)$, $k = 1, 2, \ldots, 7$, $(R = 7)$.

For this example, the desired state y^d corresponds to

$$\vec{u}^d = (0.8, 0.1, -0.3, 0, 0.1, -0.3, 0).$$

Furthermore, we chose $\Omega_{obs} = [0.44,\ 0.56] \times [0.44,\ 0.56]$ in the center of the domain $\Omega = [0,\ 1] \times [0,\ 1]$ and initialized the TRPOD algorithm with $u_0(t) \equiv 0.01$ and $\delta_0 = 0.25$.

Table 5.2 lists the values of the objective function, $f(u_k)$, and the model function at the beginning of each iteration k, $m(u_k)$. The trust region radius, δ_k, the computed step length, $||s_k||$, the quotient of achieved reduction to the predicted reduction, ρ_k, and the number of POD basis elements, M, at the beginning of each TRPOD iteration. Furthermore, for this example we checked if the gradient error condition (4.10)

$$\zeta_k := \frac{|(g_k - \nabla f(u_k))^T s_k|}{||g_k||\,||s_k||} \leq 1 - \eta_2 \tag{5.1}$$

is satisfied. The values of ζ_k are also shown in Tab. 5.2 so that we can realize that $\zeta_k \leq 0.25$ holds for $k \geq 1$. Based on the values of s_k, $m_k(u_k)$, $m_k(u_k + s_k)$ and g_k we found that (4.12) is satisfied with $b_k \leq c_b = 2$. We stopped the TRPOD algorithm after four iterations where the objective function value is reduced from $f(u_0) = 5.99e{-}4$ to $f(u_5) = 4.67e{-}7$.

In Fig. 6.1 the control iterates are shown and compared to u^d. Fig. 6.2 depicts u^d, u_4, and the solution of the reduced order control problem (without TRPOD), u_{opt}. Figs. 6.3 and 6.4 illustrate the improvements in the objective function for this example.

Example 2. We consider the boundary control of a cavity flow where a vortex evolves resulting from the movement of the top wall of the cavity at a constant horizontal velocity. The control action is a horizontal movement of the bottom wall of the cavity such that a second vortex evolves that counteracts the first vortex

TABLE 5.2
TRPOD-Results (Example 1)

k	$f(u_k)$	$m_k(u_k)$	δ_k	$\|\|s_k\|\|_2$	ρ_k	ζ_k	M
0	5.99658e-4	5.960310e-4	0.25	0.25	1.58	0.5355	4
1	3.16328e-4	3.195711e-4	0.5	0.46	1.51	0.1339	4
2	3.07127e-5	3.768390e-5	1	0.20	0.81	0.1183	4
3	2.60034e-6	2.454936e-6	2	0.22	0.97	0.1230	6
4	4.67446e-7						

(see Fig. 6.5). Thus, the control variable, $u(t)$, is the time-dependent magnitude of this horizontal boundary velocity. The desired state, y^d, corresponds to a predefined time-dependent bottom wall velocity, u^d, such that we are able to compare the computed optimal control to the exact solution. Due to a time discretization of $\Delta t = 0.03$ we get $n = 167$ control variables, i.e., the control space is of maximum dimension. We choose the observation domain to be the entire cavity, $\Omega_{obs} = \Omega$, and initialize the TRPOD algorithm with initial control of $u_0(t) \equiv 0.1$ (constant over time) and initial TR radius of $\delta_0 = 2$.

Table 5.3 lists the values of the objective function, $f(u_k)$, and the model function at the beginning of each iteration k, $m(u_k)$, the trust region radius, δ_k, the computed step length, $||s_k||$, the quotient of achieved reduction to the predicted reduction, ρ_k, and the number of POD basis elements, M, at the beginning of each TRPOD iteration.

In the beginning of the iterative process, the trust region constraint is active for the computed steps, and the quotient, ρ_k, indicates that the trust region radius should be further increased, keeping the same model, $m(u_k)$. However, in the TRPOD method, the state $y(u_k + s_k)$ is computed in the evaluation of the quotient, ρ_k, thus we can use this new information to update the model (instead of keeping the previous model). We stop the TRPOD algorithm after 5 iterations, where the objective function value is reduced from $f(u_0) = 0.229239$ to $f(u_5) = 0.000132$.

TABLE 5.3
TRPOD-Results (Example 2)

k	$f(u_k)$	$m_k(u_k)$	δ_k	$\|\|s_k\|\|_2$	ρ_k	M
0	0.229239	0.118274	2	2.00	2.36	5
1	0.149097	0.148143	4	4.00	1.27	5
2	0.080427	0.080532	8	6.84	1.47	5
3	0.006864	0.008835	16	2.87	0.93	6
4	0.000246	0.000795	32	0.29	3.61	8
5	0.000132					

Fig. 6.6 depicts the control iterates, u_k, and compared to the desired optimal control, u^d. Fig. 6.7 depicts u^d, u_5, and the solution of the reduced order control problem without TR modifications, u_{opt}. In the case without TRPOD we used a single POD based control model, corresponding to the initial control u_0, without imposing a TR constraint. Figs. 6.8 and 6.9 illustrate the improvements in the objective function.

Fig. 6.10 depicts vector plots of the first four POD basis functions corresponding to the initial control, and Fig. 6.11 depicts similar plots corresponding to the computed optimal control. These figures illustrate the significant change in the POD basis due to the changes in the control input.

6. Discussion and Concluding Remarks. We present a robust, globally convergent, approach to optimal control based on POD modeling. This approach can be interpreted in the context of trust region methods using general nonlinear model functions with inexact gradient information. Convergence results for this class of trust region methods carry over to the TRPOD method. Numerical experiments indicate the effectiveness of this approach. Optimization without POD reduced order models requires the solution of the full state equation and full sensitivity (or adjoint) equations for each 'value' of the control during the optimization process. Using POD based control models we only have to solve the full state equation if we intend to build a new model. Then we can perform a sequence of optimization steps using reduced order gradient information until we have to update the model. This amounts in saving a lot of computational work if the solution of the full state equation (and full sensitivity or adjoint equations) is computationally expensive.

The numerical results also demonstrate that the use of POD for reduced order optimal control, based on control input that is far from the solution, may lead to a large error in the approximated solution.

Unlike the traditional TR theory (see e.g. [2, 16]), we have $m_k(u_k) \neq f(u_k)$. Still, we do not need to change most of the conventional TR convergence theory, since the quality of the approximation of $f(u_k)$ by $m_k(u_k)$ does not enter into the proofs directly. However, we have to guarantee through condition (4.10) that the gradient of the model function, ∇m_k, approximates the gradient of the 'full' objective function, ∇f, sufficiently well. In general we do not expect the model gradient, ∇m_k, to be a good approximation of the gradient, ∇f, unless sensitivity (or adjoint) information of the 'full' problem is taken into account. In this context, it is still not clear, e.g., how to incorporate snapshots of the sensitivities into the POD model without compromising the quality of the state approximation.

REFERENCES

[1] K. Afanasiev and M. Hinze, *Adaptive control of a wake flow using proper orthogonal decomposition*, Preprint No. 648/1999, Technische Universität Berlin, Berlin, 1999.

[2] N. Alexandrov, J.E. Dennis, Jr., R.M. Lewis and V. Torczon, *A trust region framework for managing the use of approximation models in optimization*, ICASE Report 97-50, ICASE, NASA Langley Research Center, Hampton, 1997.

[3] J.A. Atwell and B.B. King, *Proper orthogonal decomposition for reduced basis feedback controllers for parabolic equations*, ICAM Report 99-01-01, Virginia Polytechnic Institute and State University, Blacksburg, 1999.

[4] A.K. Bangia, P.F. Batcho, I.G. Kevrekidis and G.E. Karniadakis, *Unsteady two-dimensional flows in complex geometries: Comparative bifurcation studies with global eigenfunction expansions*, SIAM J. Sci. Comput., **18**, No. 3 (1997), pp. 775-805.

[5] H.T. Banks, G.M. Kepler and H.T. Tran, *Reduced order model compensator control of species transport in a CVD reactor*, Technical Report CRSC-TR99-15, North Carolina State University, Raleigh, 1999.

[6] G. Berkooz, P. Holmes and J. Lumley, *The proper orthogonal decomposition in the analysis of turbulent flows*, Ann. Rev. Fluid Mech, **25** (1993), pp. 777-786.

[7] A. Booker, J.E. Dennis, Jr., P.D. Frank, D.B. Serafini, V. Torczon and M.W. Trosset, *A rigorous framework for optimization of expensive functions by surrogates*, ICASE Report 98-47, ICASE, NASA Langley Research Center, Hampton, 1998.

[8] R.G. Carter, *On the global convergence of trust region algorithms using inexact gradient information*, SIAM J. Num. Anal., **28**, No. 1 (1991), pp. 251-265.

[9] K. Ito and S.S. Ravindran, *A reduced order method for simulation and control of fluid flows*, J. Comput. Phys., **143**, No. 2, (1998), pp. 403-425.

[10] K. Kunisch and S. Volkwein, *Control of the Burgers' equation by a reduced order approach using proper orthogonal decomposition*, J. Optimization Theory Appl., **102**, No. 2 (1999), pp. 345-371.

[11] H.V. Ly and H.T. Tran, *Modeling and Control of physical processes using proper orthogonal decomposition*, Technical Report CRSC-TR98-37, North Carolina State University, Raleigh, 1999.

[12] J.J. Moré, *Recent developments in algorithms and software for trust region methods*, in A. Bachem, M. Grötschel and B. Korte (eds.), Mathematical programming: The state of the art, Springer, 1983.

[13] S.S. Ravindran, *Proper orthogonal decomposition in optimal control of fluids*, technical memorandum TM-1999-209113, NASA Langley Research Center, Hampton, 1999.

[14] L. Sirovich, *Turbulence and the dynamics of coherent structures, Part I: Coherent structures*, Quart. Appl. Math., **45**, No. 3 (1987), pp. 561-571.

[15] K.Y. Tang, W.R. Graham and J. Peraire, *Active flow control using a reduced order model and optimum control*, AIAA, Nr. 1996-19-46, 1996.

[16] P.L. Toint, *Global convergence of a class of trust region methods for nonconvex minimization in Hilbert space,* IMA J. Num. Anal., **8**, No. 2 (1988), pp. 231-252.

[17] S. Turek, *FEATFLOW - Finite element software for the incompressible Navier-Stokes equations: User manual*, University of Heidelberg, Heidelberg, 1995.

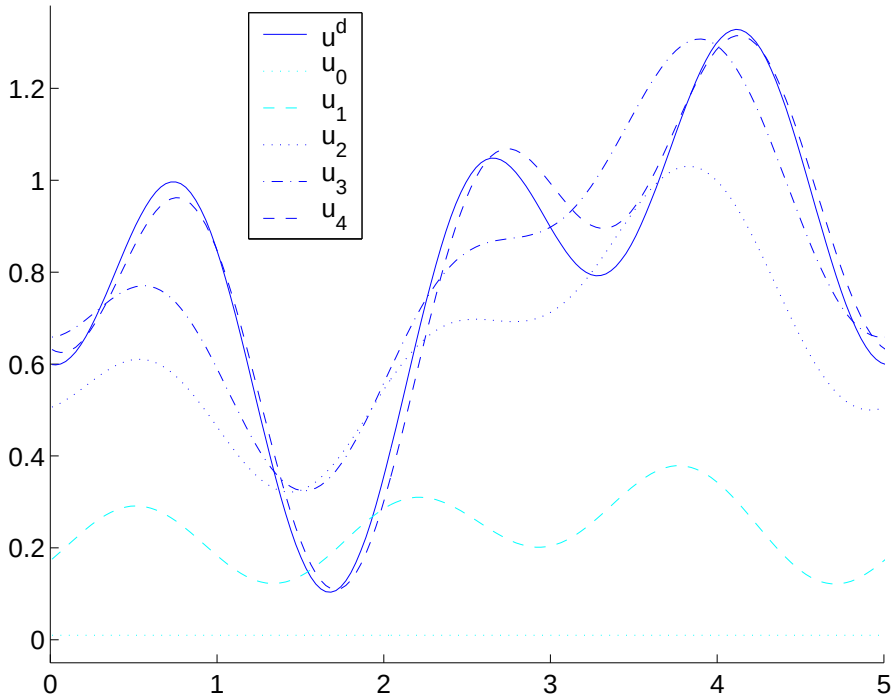

FIG. 6.1. *Control iterates u_k (Example 1)*

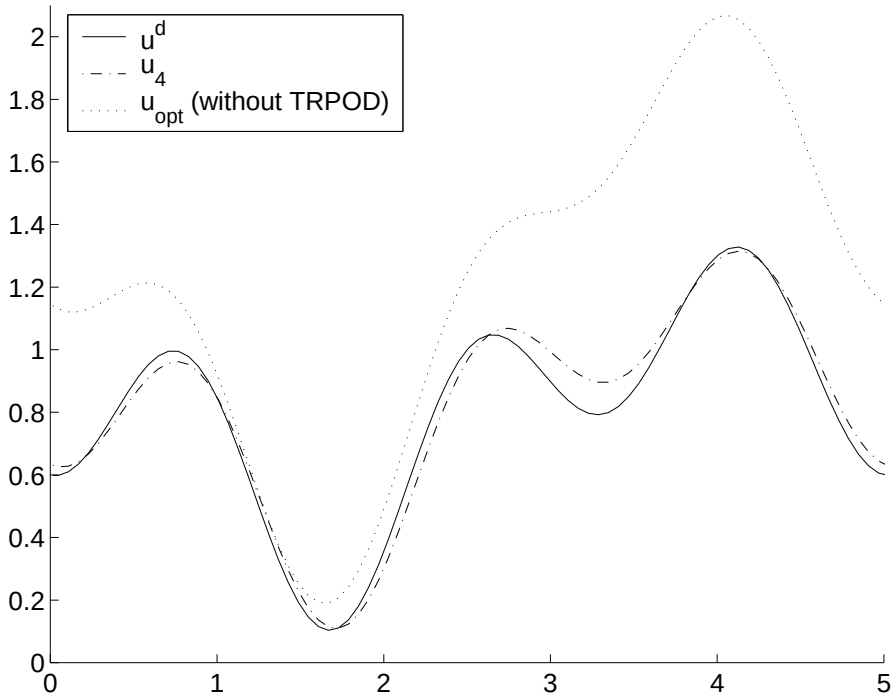

FIG. 6.2. *Comparison of Controls (Example 1)*

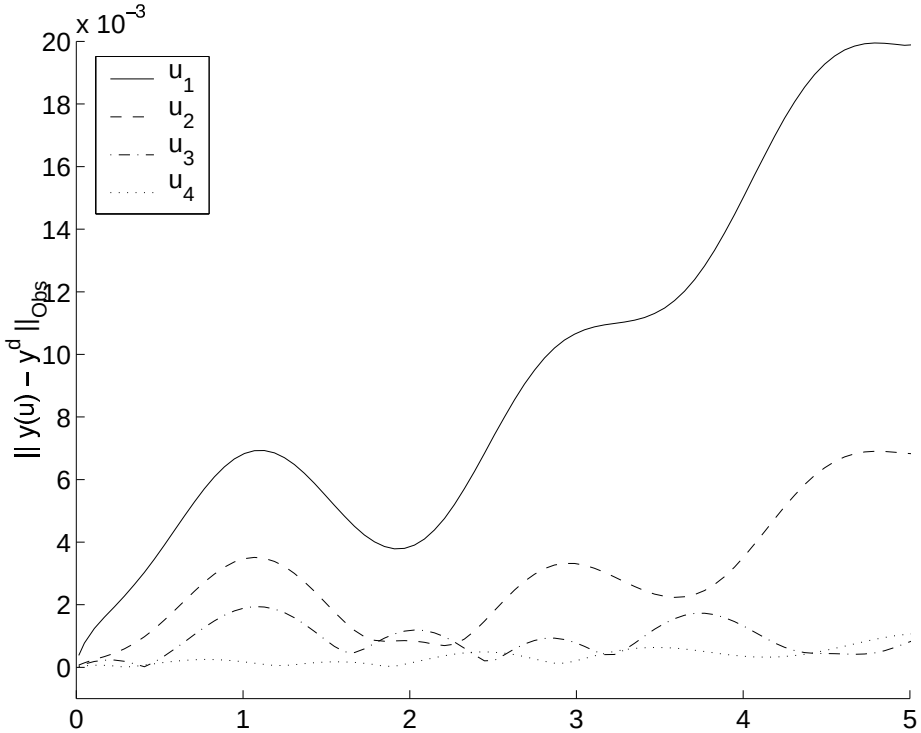

FIG. 6.3. $||y(u) - y^d||$ *for iterates* u_k *(Example 1)*

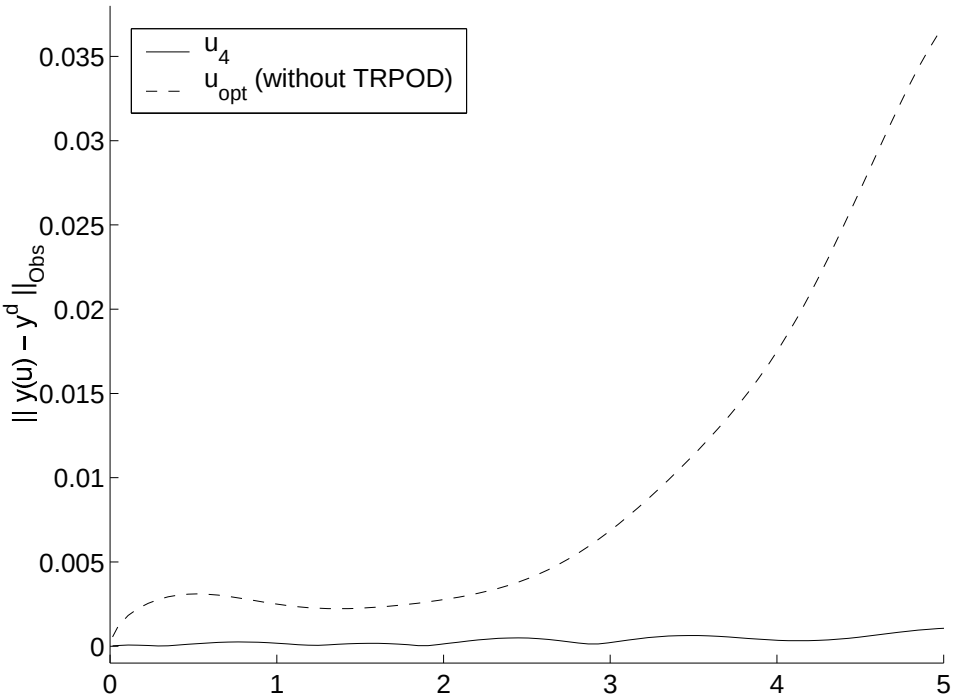

FIG. 6.4. *Comparison of* $||y(u) - y^d||$ *(Example 1)*

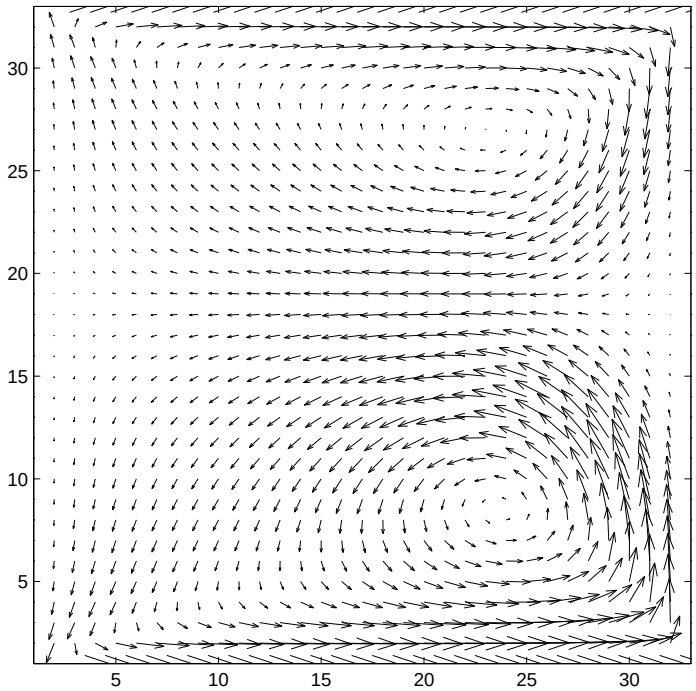

FIG. 6.5. *Control Configuration (Example 2)*

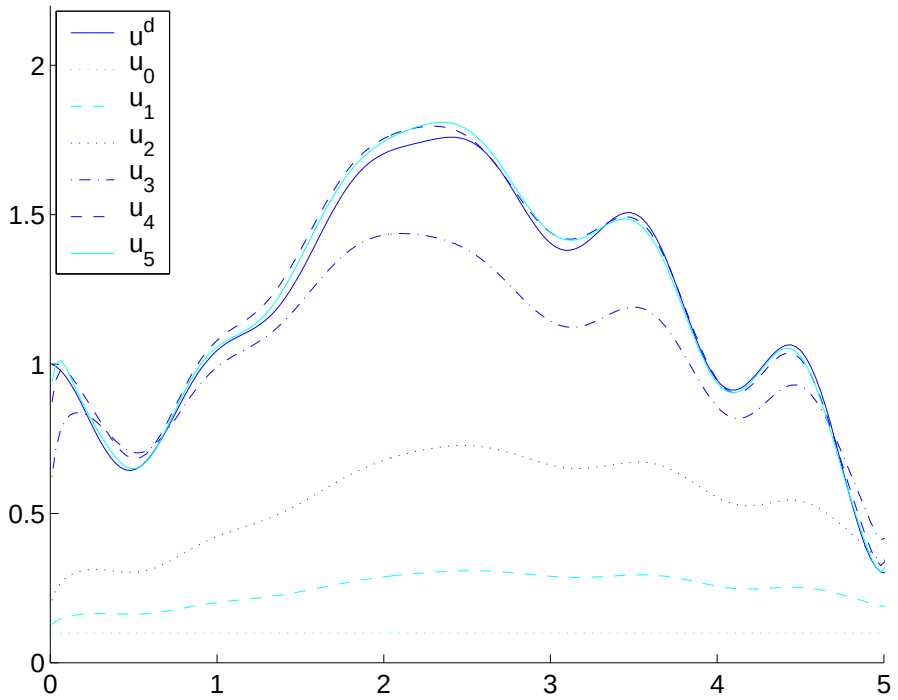

FIG. 6.6. *Control iterates u_k (Example 2)*

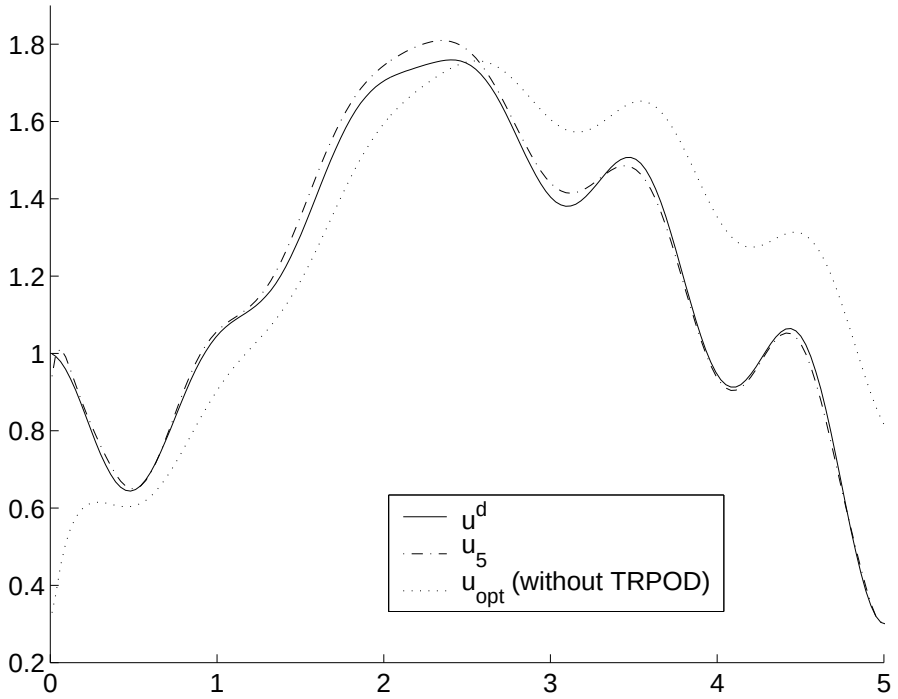

FIG. 6.7. *Comparison of Controls (Example 2)*

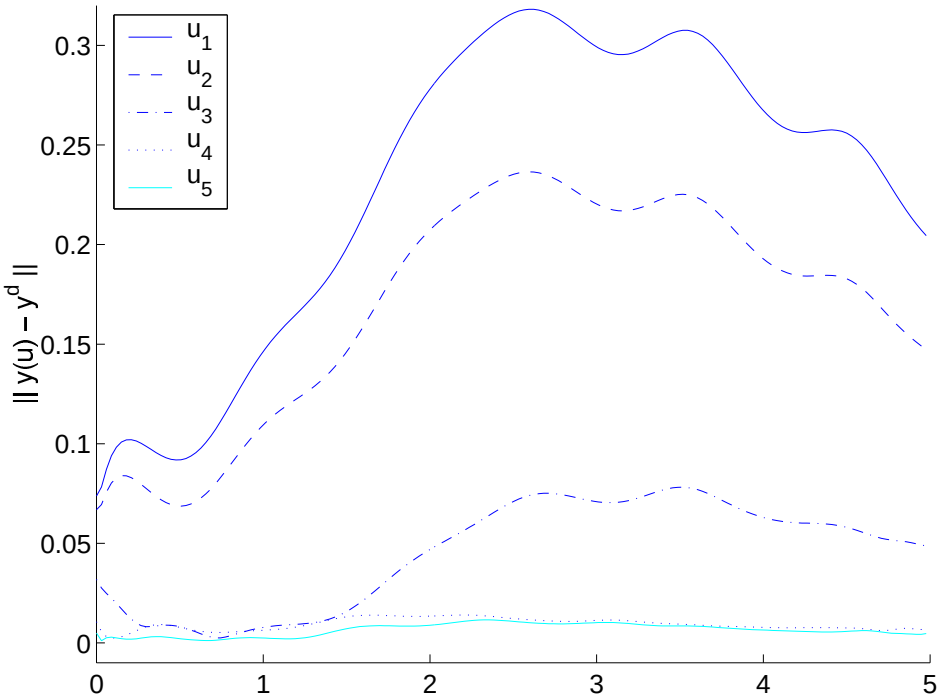

FIG. 6.8. $||y(u) - y^d||$ *for iterates* u_k *(Example 2)*

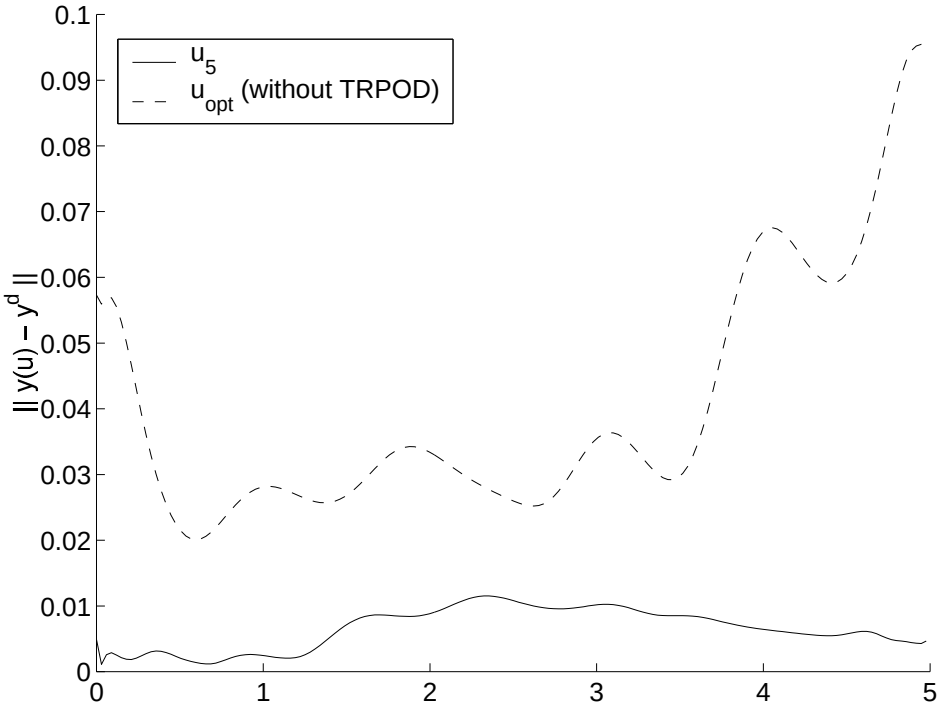

FIG. 6.9. *Comparison of* $||y(u) - y^d||$ *(Example 2)*

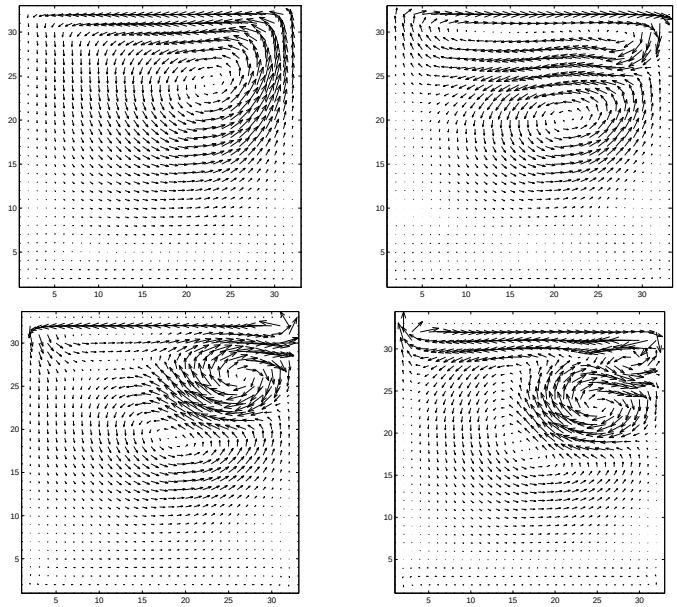

FIG. 6.10. *POD basis functions* $\psi^1 - \psi^4$ *corresponding to* u_0 *(Example 2)*

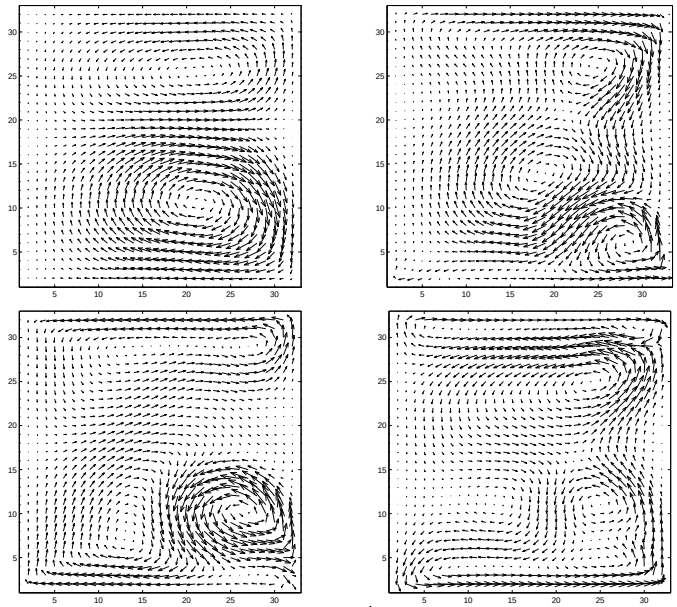

FIG. 6.11. *POD basis functions* $\psi^1 - \psi^4$ *corresponding to* u_5 *(Example 2)*