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Abstract

This paper presents a control method based on virtual passive dynamic control that will stabilize a robot manipulator using joint torque sensors and a simple joint model. The method does not require joint position or velocity feedback for stabilization. The proposed control method is stable in the sense of Lyaponov. The control method was implemented onseveral joints of a laboratory robot. The controller showed good stability robustness to system parameter error and to the exclusion of nonlinear dynamic effects on the joints. The controller enhanced position tracking performance and, in the absence of position control, dissipated joint energy.

I. Introduction

Control of robotic systems has been a difficult problem due to the nonlinearity of the complex system equations. Several techniques to control this nonlinear system have been studied. Some methods, such as Jacobian linearization[1] and pseudolinearization[2], have attempted to linearize the system and apply classical linear system control techniques to the problem. These methods are only valid close to a linearization point or trajectory. The feedback linearization[3] technique attempts to linearize the equations over a large workspace and has been popular in the robotic control literature for some time. Feedback linearization requires good knowledge of the system parameters and states, else some of the nonlinearity will not be canceled out. Variable structure control is a purely nonlinear control method which has been a popular research topic[4]. Although variable structure control is robust, the fast switching required by the controller to maintain this robustness is difficult to achieve without chatter. These are only a few of the many control methods that have been applied to robot systems.

The main reason for the nonlinearity in the equations is the need to calculate the dynamic effects on the structure. Controlling the nonlinear plant based on the full dynamic equations is not the only method for controlling robot manipulator. If the joints have a torque sensor along the drive axis, the problem can be reduced to controlling the individual joint dynamics. Work in this area has been published recently[5,6]. These methods use known, mostly linear, electric motor driven joint models to control joint acceleration and velocity. These methods still require measurement of joint position and velocity to compute the control inputs.

Passivity based control methods have been applied to control in robotics [7,8] and vibration control of space structures[9]. The problem with the basic passivity control approach is that it requires velocity feedback. The virtual passive dynamic control approach [10] has been successful in stabilizing systems with displacement, velocity, or a combination of acceleration with displacement and velocity feedback.

This paper develops a method to stabilize a robot manipulator with joint torque sensors without directly measuring joint position, velocity, or acceleration. The torque sensor output will be used in conjunction with a simple joint model and the virtual passive dynamic based control technique to quickly dissipate the kinetic energy in the robot system. The robustness of the system will be discussed and the results of an experiment in which a robot joint was controlled using the proposed method will be shown.

II. Dynamic model

The following derivation is based on a model of a direct drive, electric motor driven, revolute joint with an output torque sensor presented in Kosuge[5].

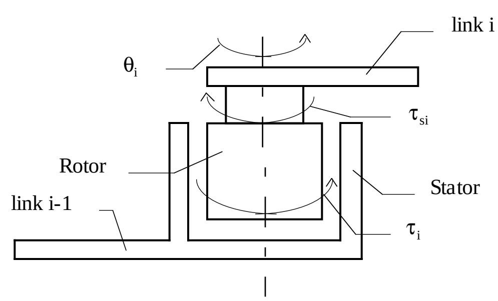

Figure 1: Diagram of proposed direct drive joint

This joint is assumed to be part of a serial linkage consisting of other revolute joints and rigid links. The structure of the proposed joint is shown in Fig. 1. Each joint is assumed to have two parameters, rotor inertia and viscous friction.

Paraphrasing the development in Kosuge [5], the model of the joint is determined by applying a Newton-Euler iterative dynamics[11] approach to a rigid, serially linked structure with revolute joints. This derivation includes the motor inertia and viscous friction terms. The full equation for the joint torque given an arbitrary trajectory was computed. The terms required to compute the torque were divided into three groups: terms which depend only on the i^{th} joint, terms depending on other joints which affect the i^{th} joint, and terms that involve link inertias, masses, and lengths. The joint torque model resulting from this derivation is,

$$\tau_i = m_i\ddot{\theta}_i + \tau_{si} + v_i\dot{\theta}_i + f_i \tag{1}$$

$$M_{ri} \equiv A_{i-1}^0 M_i A_0^{i-1} \tag{2}$$

$$m_i \equiv z_0^T M_{ri} z_0 \tag{3}$$

$$\omega_i = \omega_{i-1} + z_{i-1}\dot{\theta}_i \tag{4}$$

$$\dot{\omega}_i = \dot{\omega}_{i-1} + z_{i-1}\ddot{\theta}_i + \omega_{i-1} \times (z_{i-1}\dot{\theta}_i) \tag{5}$$

$$\begin{aligned} f_i \equiv\ & z_0^T [M_{ri}(A_{i-1}^0\dot{\omega}_{i-1}) + A_{i-1}^i(A_i^0\omega_i) \\ & \times (A_i^{i-1}M_{ri}A_{i-1}^i A_i^0\omega_i) + M_{ri}(A_{i-1}^0\omega_{i-1} \times z_0\dot{\theta}_i)] \end{aligned} \tag{6}$$

where,

M_i	inertia matrix of the rotor of the i^{th} joint
v_i	coefficient of the viscous friction torque acting on the rotor around the joint axis z_{i-1}
A_j^i	rotation matrix from frame i to frame j (same origin)
M_{ri}	rotor inertia matrix in reference frame
τ_i	torque at joint input
τ_{si}	the sensed torque along the z axis at the joint output
θ_i	joint position
ω_i	angular velocity of i^{th} frame in base coordinates
z_i	z vector for i^{th} frame in the base coordinates

Joint specific terms are evident in the model while link specific terms are contained in the torque measurement along the axis of rotation. With the exception of the nonlinear term f_i, the elements of Eq. (1) depend only on values of the i^{th} joint.

In later sections, the goal will be to develop a controller that will produce u_i, the control input, that will stabilize a joint described by Eq. (1). If $\tau_i=u_i$, then Eq. (1) can be solved in terms of the sensed torque.

$$\tau_{si} = -m_i\ddot{\theta}_i - v_i\dot{\theta}_i - f_i + u_i \tag{7}$$

where

$$\tau_{si} = \tau_{xi} - f_i \tag{8}$$

and

$$\tau_{xi} = -m_i\ddot{\theta}_i - v_i\dot{\theta}_i + u_i \tag{9}$$

In section IV, exact knowledge of the term τ_{xi} is assumed to derive a stabilizing control input for the system. In section V, the robustness of the system to inexact knowledge of τ_{xi} is discussed.

III. Virtual passive dynamic control

The controller design technique used in section IV is similar to the passive dynamic control technique presented in Juang[10]. This technique is based on the concept that a mechanical system can be represented by a second-order system with inertia, damping and stiffness related parameters. An active feedback controller can be designed with its dynamics equivalent to a mechanical system. The resulting controller is,

$$H_M\ddot{x}_c + H_D\dot{x}_c + H_K x_c + g(y_s) = u \tag{10}$$

where y_s is the measured system output, g is a user defined function, x_c is the controller state vector of dimension n_c, and H_M, H_D, and H_K are the controller mass, damping, and stiffness matrices respectively. These matrices are design parameters and can be chosen to meet performance and stability requirements. The function g is an arbitrary function of the measured system output, y_s. These outputs can be system states or combinations of system states.

The Lyaponov proof of the stability of the chosen control law will depend on the rate of change of the system kinetic energy. Let T be the total kinetic energy of a mechanical system (linear or nonlinear) with p control actuators at p physical locations described by p generalized coordinates x_{ai} and p control inputs u_i, $i=1,2,...,p$. These generalized coordinates and their derivatives are physical quantities of the system. If a mechanical system is holonomic and scleronomic (no explicit time dependence), the time derivative of the total kinetic energy is related to the applied forces by,

$$\frac{dT}{dt} = u^T\dot{x}_a \tag{11}$$

where $x_a = (x_{a1},x_{a2},...,x_{ap})^T$.

Choose the Lyaponov candidate function to be of the type:

$$L = T + q(x_a, \dot{x}_a, x_c, \dot{x}_c) \tag{12}$$

where q is an arbitrary function of the actuator and controller states, x_c, and their rates. Taking the time derivative and substituting Eq. (11) yields

$$\frac{dL}{dt} = u^T\dot{x}_a + \dot{q}(x_a, \dot{x}_a, \ddot{x}_a, x_c, \dot{x}_c, \ddot{x}_c) \tag{13}$$

If the control inputs, u_i, are designed to cause the rate of change of the Lyaponov function to be negative, the stability of the system is guaranteed by Lyaponov stability theory. This stability implies a continual decrease in the kinetic energy of the system.

Remembering that u is a function of system outputs, states, and controller states, it can be designed to cancel out and combine with terms in the derivative of the Lyaponov candidate function to result in an equation of the following form,

$$\frac{dL}{dt} = -\dot{x}_a^T D\dot{x}_a - \dot{x}_c^T R(x_a, \dot{x}_a, \ddot{x}_a, x_c, \dot{x}_c, \ddot{x}_c) \qquad (14)$$

where R is a linear function and D is a matrix involving the system damping. The desired function which implies a constantly decreasing Lyaponov function is,

$$\frac{dL}{dt} = -\dot{x}_a^T D\dot{x}_a - \dot{x}_c^T D_c \dot{x}_c \qquad (15)$$

that can be obtained by making the equality:

$$R(x_a, \dot{x}_a, \ddot{x}_a, x_c, \dot{x}_c, \ddot{x}_c) = D_c \dot{x}_c \qquad (16)$$

The controller state can be calculated using Eq. (16) and substituted into Eq. (10) to determine the u_i required to maintain the Lyaponov stability condition.

IV. Torque feedback

This section will describe a controller using torque sensor feedback to stabilize a robot system. Let the quantity τ_{xi} be known exactly. Inexact knowledge of τ_{xi} and its relationship to τ_{si} will be dealt with in section V. Let:

$$M_{rz} = diag(m_1, ..., m_p) \qquad (17)$$

$$V = diag(v_1, ..., v_p) \qquad (18)$$

$$\tau_x = diag(\tau_{x1}, ..., \tau_{xp}) \qquad (19)$$

Using the virtual passive controller design technique from the previous section, a controller that satisfies the Lyaponov stability criteria, (the Lyaponov stability proof may be found in Appendix A) is given by,

$$\begin{bmatrix} \dot{x}_c \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} 0 & I \\ -(R_\tau M_{rz})^{-1} K_c & -(R_\tau M_{rz})^{-1} D_c \end{bmatrix} \begin{bmatrix} x_c \\ \dot{x}_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ M_{rz}^{-1} & -M_{rz}^{-1} \end{bmatrix} \begin{bmatrix} \tau_x \\ u' \end{bmatrix} \qquad (20)$$

$$u = \begin{bmatrix} K_c & R_\tau V + D_c \end{bmatrix} \begin{bmatrix} x_c \\ \dot{x}_c \end{bmatrix} \qquad (21)$$

where D_c, R_τ, and K_c are design matrices. The restriction on these design matrices is that D_c and R_τ must be a symmetric and positive-definite . The current control input, u', is used to calculate the next control input. Note that as described in the virtual passive controller discussion, the terms x_c and $\dot{x}_c$ are not the joint position and velocity. They are controller states used to satisfy the stability condition. As a result, this controller design can dissipate joint kinetic energy without joint position or velocity feedback.

V. Stability robustness

The previous control law concentrated on a non-directly measured value, τ_{xi}, instead of the directly measured term, τ_{si}. It was also assumed that no modeling errors were present and that the nonlinear term, f_i, was calculable. If the joint in question does not have a position sensor and/or the controller did not have information from other joints, the nonlinear term is incalculable. If additive modeling errors are present, the joint model becomes:

$$u_i = (m_i + \overline{m}_i)\ddot{\theta}_i + (v_i + \overline{v}_i)\dot{\theta}_i + \tau_{si} + f_i \qquad (22)$$

Reformatting and solving for τ_{si},

$$\tau_{si} = \tau_{xi} - e_i \qquad (23)$$

$$e_i = \overline{m}_i \ddot{\theta}_i + \overline{v}_i \dot{\theta}_i + f_i \qquad (24)$$

From Eq. (24), it can be seen that if the kinetic energy in the joint declines, then the affects of the additive error terms are reduced. The magnitude of these errors should never be very high because the m_i term, which represents the rotor inertia along the z axis, should be known precisely from the motor manufacturer and v_i, the viscous friction term, while not easily modeled precisely can be closely approximated by a simple linear model. Since higher frictional forces enhance the dissipation of energy, stability will not be affected if the modeled friction is less than the actual friction. Due to the low relative magnitude of these errors, they can be considered disturbances and do not affect the overall stability of the system.

The nonlinear term f_i can also be shown to decline with kinetic energy since it is related to link angular velocity and acceleration. Assuming a serial robot with a fixed base, the first joint's angular velocity and acceleration depend on the magnitude of the first motor's velocity and position. The second joint's angular velocity and acceleration depend on the magnitude of the first and second motor's velocity and acceleration. With the fixed base assumption, the f_i term for the first joint is zero. The first joint's passive controller will then dissipate its energy decreasing the nonlinear effect on the second joint. The second joint's nonlinear term is now only dependent on its state and, for reasonable robot moves, can be treated as a disturbance. This chain can be continued for n joints.

VI. Position control

The virtual passive torque controller can be used with a position controller as shown in Fig. 2. In this example, position tracking was implemented by a PD controller whose torque output was subtracted from the sensed torque to offset the controller input. This offset input appeared to the controller as movement. The passive controller's efforts to dissipate the energy caused by this "movement" causes the joint to move in the desired direction.

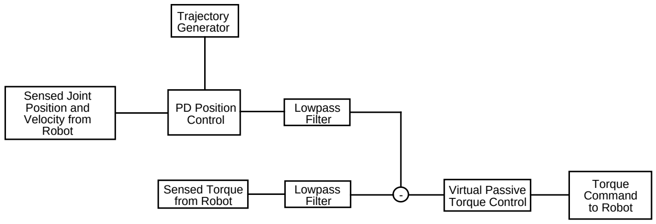

Figure 2: Control block diagram

Stability is no longer guaranteed when using the position controller in this manner. Although the torque controller can be tuned to be stable in areas of minor position controller instability at the cost of reduced position tracking elsewhere, true position controller instability will case the system to become unstable. Interaction between the position and torque controller is discussed in the next section.

VII. Experiment

To evaluate the performance of the proposed controller, it was implemented on the three wrist joints of a laboratory robot. The robot used was a Robotics Research Corporation (RRC) 807i manipulator. The 807i has 7 degrees of freedom, is 0.8m long, and has an approximate payload of 10kg. The manipulator is shown in Fig. 3.

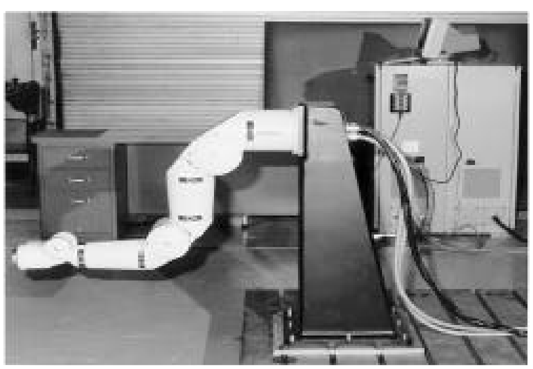
Figure 3: Robotics Research 807i

The goal of the experiment is to show:

1. The controller enhances tracking performance in the presence of unknown end-effector loading
2. The controller dissipates joint energy without a position command
3. The controller is stable in multiple configurations

The virtual passive controller shown in Fig. 2 was implemented on a 68040 based computer, called the control processor, which communicated with the robot controller across a bus-to-bus interface. The control processor sent torque commands to the robot at 200Hz.

The RRC robot was commanded in torque mode. In this mode, the robot controller uses a basic torque controller to overcome joint effects. Its goal is to make the harmonic gear driven joints perform like direct drive joints. The resulting system does reduce geartrain effects but does not eliminate the effects. The virtual passive controller must handle any remaining geartrain friction, hysteresis, or backlash. Although the model presented in section II was direct drive, the controller can be used on gear driven joints if the effects geartrain friction and the gear ratio are considered and nonlinear geartrain effects, such as backlash, are minimized.

The motor parameters required by the controller were not available from the robot manufacturer. Identifying the parameters of motors installed in the robot proved difficult due to the parameters small size and the inability to bypass the low level torque controller. As a result, qualitative data was used to estimate the parameters and the gains chosen to produce the required performance. The resulting performance with substantial modeling errors shows the stability robustness of the controller.

Tracking performance

To examine how tracking performance varies with end-effector loading, weights were attached to the end-effector. The weights were chosen to represent realistic loadings for a robot of this type and not saturate the actuators. The three weighting conditions were 0,5, and 10lbs. The weights were attached to the end-effector with a beam which offset the weight by approximately 23cm from the center of the end-effector to load the wrist joints properly.

Two controller configurations were used. The first, utilizing virtual passive control, was the same as found in Fig. 2. The second, not utilizing virtual passive controller, sent the output of the PD controller directly to the robot bypassing the virtual passive controller. The PD controller was tuned to give reasonable performance without the virtual passive controller . The same PD gains were used for both controller configurations.

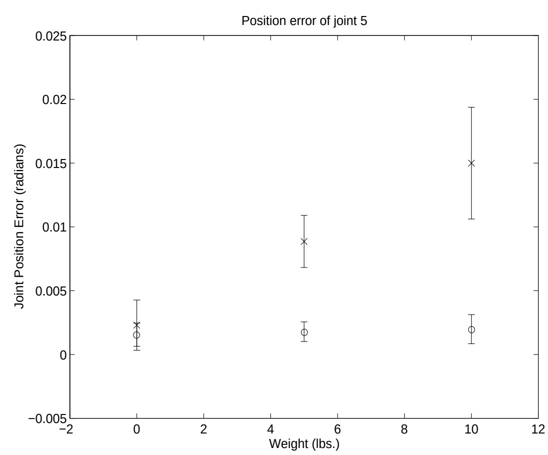

Figure 4: Error in joint position with and without virtual passive controller at different weights (the o plots are with and the x plots are without the passive controller)

Figure 4 shows the mean and standard deviation of the error of joint 5, the wrist roll joint, along a representative trajectory. All seven joints were actuated on this trajectory (the lower four joints using the RRC position controller) to add a dynamic effect to the measurements. Using the virtual passive controller, the torque sensors compensated for the added loading maintaining a small error mean and fairly constant standard deviation. Without the passive controller, the PD controller error mean and standard deviation increases as the weight increases.

Energy dissipation

Without the position control generated torque offset, the theory states that the controller should attempt to dissipate joint energy. With proper controller tuning, this dissipation should lead to the joint stopping and resisting movement with the virtual passive controller enabled. To test this hypothesis, the robot was commanded along a trajectory actuating all robot joints. Three seconds into the trajectory, the position control torque offset was removed. Figure 5 shows the response of joint 7, the toolplate roll joint, along a representative trajectory. As shown, the controller quickly stops the joint. The small difference in steady state position is due to the differing weights. Given enough force, the robot joint can be pulled off the final position. When the force is lowered, it will remain at a new position close to the position where it was when the force was removed.

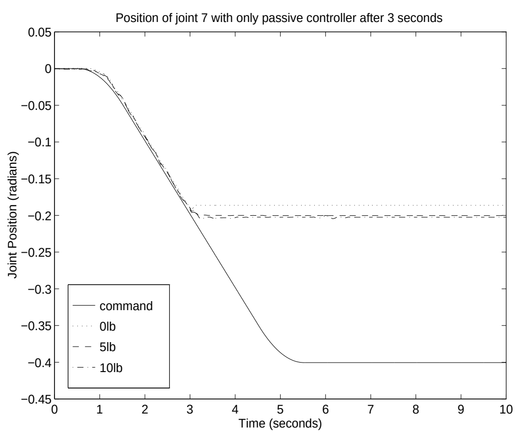

Figure 5: Position response of joint 7 with position control offset removed at 3 sec.

The resulting controller/joint combination acts as a highly damped joint. Without the virtual passive controller, the joint runs quickly to the joint stop. It should be noted that having this highly damped property does not ruin position tracking as is could if the joint mechanism itself was highly damped.

The ability of the controller to dissipate energy has applications in fault tolerant control. This controller can act as an active joint brake in the absence of a physical brake. It does not rely on joint position or velocity feedback for stability so it is robust to the failure of these sensors.

Controller stability

The virtual passive controller was tuned to be stable in different joint configurations and loadings. When the virtual passive gains were increased to beyond those used in the experiment which could produce the quick energy dissipation shown in Fig. 5, two stability problems arose.

The first stability problem involved controller oscillation. If one joint's torque controller gains are too high for the current joint load it can begin to oscillate. As the load is increased, the oscillation damps out. This oscillation can induce an oscillation into other joints otherwise stable virtual passive torque controllers. The oscillations can be eliminated by lowering the passive torque controller gains.

The seconds stability problem involves the interaction of the position and virtual passive torque controllers. Figure 6 shows the sensed torque response of joint 5 to a trajectory with the position controller disabled at three seconds. A large torque vibration occurred in joints 5 and 7 at the 5lb weight condition. Joint 5 and 7 are both roll joints whose axes were becoming more parallel as the trajectory continues. The parallel condition allowed a vibration started in one joint to resonate into the other joint's sensor. This resonance does not occur if the passive controller is bypassed on one of the two joints. The resonance continues if the passive controller on joint 6 is bypassed removing it as a cause. The interesting point of the data is that the resonance occurred at the middle weight condition, not the high or low condition. This can be explained by the relative control authority of the passive controller. At low weights, the passive controller had little authority and the control is mostly done by the position controller. As the weight increases, the passive controller gains more authority. At a moderate weighting condition, the passive controller's control authority is similar to the position controller and they begin to interact with neither controller dominating. At higher weighting conditions, the passive controller begins to dominate the position controller. When the position controller's bias is zeroed at three seconds, the passive controller quickly damps out the resonance. The position response resulting from these torques was not adversely affected. However, these quick torque changes put undue wear on the drivetrain and can excite modes in robot payloads. Further refining of position controller and passive torque controller gains will alleviate the problem.

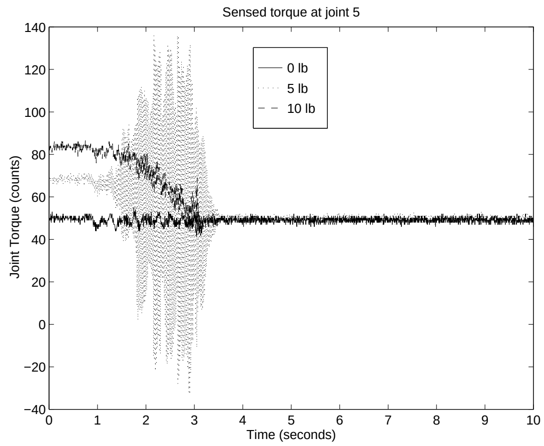

Figure 6: Torque instability caused by controller interaction

VIII. Conclusion

This paper presented a robot joint controller design using virtual passive control with a joint torque sensor. A Lyaponov stable control law was presented and experimentally tested. Given reasonable joint torque sensor data, the system provides good stabilization performance with parameter errors and treating nonlinear dynamic effects as a disturbance. By using the torque sensor data, manipulator link dynamics and loads do not need to be modeled. The controller does not require joint position or velocity feedback to dissipate joint kinetic energy.

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Appendix A: Lyaponov stability proof

The goal of this appendix is to provide a Lyaponov stability proof for using virtual passive dynamic control with torque sensor feedback to stabilize a system. For simplicity of the proof, let the quantity τ_{xi} be known exactly. Choose the Lyaponov function similar to Eq. (12) with $x_a = \theta$. A candidate Lyaponov function for the stability proof is:

$$L = T + \frac{1}{2}(\dot{x}_a + \dot{x}_c)^T K_\tau M_{rz}(\dot{x}_a + \dot{x}_c) + \frac{1}{2} x_c^T K_c x_c \tag{A1}$$

$$M_{rz} = diag(m_1, ..., m_p) \tag{A2}$$

Where K_τ and K_c are design matrices and M_{rz} represents the rotor inertia along the z axis of joints in the system. Differentiating (A1),

$$\frac{dL}{dt} = u^T \dot{x}_a + (\ddot{x}_a + \ddot{x}_c)^T K_\tau M_{rz}(\dot{x}_a + \dot{x}_c) + \dot{x}_c^T K_c x_c \tag{A3}$$

Now, select the control input u to be a function of τ_{xi}, the controller state vector, the rotor inertia, and the controller design matrix K_τ such that

$$u = K_\tau \tau_x - K_\tau M_{rz} \ddot{x}_c + K_\tau V \dot{x}_c \tag{A4}$$

$$V = diag(v_1, ..., v_p) \tag{A5}$$

$$\tau_x = diag(\tau_{x1}, ..., \tau_{xp}) \tag{A6}$$

where τ_{xi} is defined in Eq. (9). Substituting the equation for τ_x into Eq. (A4) yields

$$u = -K_\tau M_{rz}(\ddot{x}_a + \ddot{x}_c) - K_\tau V(\dot{x}_a - \dot{x}_c) + K_\tau u \tag{A7}$$

Let,

$$R_\tau = (I - K_\tau)^{-1} K_\tau \tag{A8}$$

Substituting (A8) into (A7),

$$u = -R_\tau M_{rz}(\ddot{x}_a + \ddot{x}_c) - R_\tau V(\dot{x}_a - \dot{x}_c) \quad \text{(A9)}$$

If $R_\tau V$ and $R_\tau M_{rz}$ are chosen symmetric, replacing u in the time derivative of the Lyaponov function Eq. (A3) produces:

$$\begin{aligned}\frac{dL}{dt} &= -(\ddot{x}_a + \ddot{x}_c)^T R_\tau M_{rz}\dot{x}_a - (\dot{x}_a - \dot{x}_c)^T R_\tau V\dot{x}_a \\ &\quad + (\ddot{x}_a + \ddot{x}_c)^T R_\tau M_{rz}(\dot{x}_a + \dot{x}_c) + \dot{x}_c^T K_c x_c\end{aligned} \quad \text{(A10)}$$

Canceling terms and reformatting,

$$\begin{aligned}\frac{dL}{dt} &= -\dot{x}_a^T R_\tau V\dot{x}_a + \dot{x}_c^T R_\tau V\dot{x}_a \\ &\quad + \dot{x}_c^T R_\tau M_{rz}(\ddot{x}_a + \ddot{x}_c) + \dot{x}_c^T K_c x_c\end{aligned} \quad \text{(A11)}$$

or

$$\begin{aligned}\frac{dL}{dt} &= -\dot{x}_a^T R_\tau V\dot{x}_a + \dot{x}_c^T [R_\tau V\dot{x}_a \\ &\quad + R_\tau M_{rz}(\ddot{x}_a + \ddot{x}_c) + K_c x_c]\end{aligned} \quad \text{(A12)}$$

Now, define the following equality

$$R_\tau V\dot{x}_a + R_\tau M_{rz}(\ddot{x}_a + \ddot{x}_c) + K_c x_c = -D_c\dot{x}_c \quad \text{(A13)}$$

where D_c is a symmetric and positive-definite matrix. This equality produces a set of equations that calculate the controller state vector as a function of the sensed torque. Substituting Eq. (A13) into Eq. (A12), the time derivative of the Lyaponov function becomes:

$$\frac{dL}{dt} = -\dot{x}_a^T R_\tau V\dot{x}_a - \dot{x}_c^T D_c\dot{x}_c \quad \text{(A14)}$$

This function results in a constantly decaying Lyaponov function if $R_\tau V$ and D_c are chosen to be positive definite and x_c is calculated using a reformatted version of Eq. (A13),

$$R_\tau M_{rz}\ddot{x}_c + D_c\dot{x}_c + K_c x_c = -R_\tau(M_{rz}\ddot{x}_a + V\dot{x}_a) \quad \text{(A15)}$$

Since only τ_x is known, substitute Eq. (9) and Eq. (A6) into Eq. (A15) and add $R_\tau u$ to both sides to replace dependence on measured joint acceleration and velocity with τ_x,

$$R_\tau M_{rz}\ddot{x}_c + D_c\dot{x}_c + K_c x_c + R_\tau u = R_\tau \tau_x \quad \text{(A16)}$$

Using Eq. (A16) and Eq. (A4), the control input required to stabilize the system can be calculated as:

$$\begin{aligned}u &= R_\tau \tau_x - R_\tau M_{rz}[(R_\tau M_{rz})^{-1}(-D_c\dot{x}_c - K_c x_c) \\ &\quad + M_{rz}^{-1}\tau_x)] + R_\tau V\dot{x}_c\end{aligned} \quad \text{(A17)}$$

If R_τ and M_{rz} are invertible, as they should be since R_τ can be chosen positive definite and M_{rz} is the full rank diagonal joint rotor inertia matrix, Eq. (A17) reduces to:

$$u = (R_\tau V + D_c)\dot{x}_c + K_c x_c \quad \text{(A18)}$$

Although the torque sensor related term cancels out of u, the control input is not independent of τ_x because τ_x is used to form x_c and its derivative. With τ_x as the input, u' as the current torque command, and u as the output, the control signal required to stabilize the system can be calculated using Eq. (20) and (21).