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Simplified Analytical Model of a Six-Degree-of-Freedom Large-Gap Magnetic Suspension System

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ABSTRACT

A simplified analytical model of a six-degree-of-freedom large-gap magnetic suspension system is presented. The suspended element is a cylindrical permanent magnet that is magnetized in a direction which is perpendicular to its axis of symmetry. The actuators are air core electromagnets mounted in a planar array. The analytical model consists of an open-loop representation of the magnetic suspension system with electromagnet currents as inputs.

INTRODUCTION

This paper develops a simplified analytical model of a six-degree-of-freedom (6DOF) Large-Gap Magnetic Suspension System (LGMSS). The LGMSS is a conceptual design for a ground-based experiment which can be used to investigate the technology issues associated with magnetic suspension at large gaps, such as accurate suspended element control and accurate sensing (ref. 1). This technology is applicable to future efforts which range from magnetic suspension of wind tunnel models to advanced spacecraft experiment isolation and pointing systems. The 6DOF model is an extension of the five degree of freedom (5DOF) model developed in reference 2. The suspended element is a cylindrical permanent magnet which is magnetized perpendicular to its axis of symmetry and the actuators are air core electromagnets mounted in a planar array. The electromagnet array is mounted horizontally with the suspended element levitated above the array by repulsive forces. In the nominal suspended element orientation, the axis of symmetry is horizontal also. The 5DOF model developed in reference 2 was used to investigate two LQR control approaches for an LGMSS in reference 3. In reference 3, the simplifying assumption was made that the change in field and field gradients with respect to suspended element displacements was negligible. In reference 4 the analytical model developed in reference 2 was linearized and extended to include the change in fields and field gradients with respect to suspended element displacements and the open-loop characteristics of the resulting system were investigated. Reference 5 developed the expanded equations (up to second order) for torque and force on a cylindrical permanent magnet core for two orientations of the core magnetization vector. One orientation was parallel to the axis of symmetry of the core and the other was perpendicular to this axis. In general, the higher order terms in the expanded equations can be neglected. However, in the case where the magnetization vector is perpendicular to the axis of symmetry, the expanded equations indicate that torque about the magnetization vector can be produced by controlling a second-order gradient term directly. This allows the core to be controlled in 6DOF. In this paper the 6DOF analytical model is developed by following the approach detailed in references 2 and 4 using the equations for a cylindrical permanent magnet core uniformly magnetized perpendicular to its axis of symmetry which are developed in reference 5. The analytical model consists of an open-loop representation of the magnetic suspension system with electromagnet currents as inputs.

SYMBOLS

$\mathcal{A}$ system matrix (state-space representation)

a radius of core, m

$\mathcal{B}$ input matrix (state-space representation)

B magnetic flux density vector, T

$\left[\partial\mathbf{B}\right]$ matrix of field gradients, T/m

F total force vector on suspended element, N

$\mathbf{F}_c$ magnetic force vector on suspended element, N

$\mathbf{F}_d$ disturbance force vector on suspended element, N

$\mathbf{F}_g$ gravitational force vector on suspended element, N

g acceleration due to gravity ($1g \approx 9.81 m/sec^2$), m/sec^2

h suspension height (suspended element centroid to top plane of coils), m

I coil current vector, A

$[\mathbf{I}_c]$ moment of inertia about the principal axes of the suspended element, $kg\text{-}m^2$

k_{in} constant representing magnitude of B_{in} produced by I_{max} in coil n

k_{ijn} constant representing magnitude of B_{ijn} produced by I_{max} in coil n

$k_{(ij)kn}$ constant representing magnitude of $B_{(ij)kn}$ produced by I_{max} in coil n

K_{in} k_{in}/I_{max}, T/A

K_{ijn} k_{ijn}/I_{max}, T/m/A

$K_{(ij)kn}$ $k_{(ij)kn}/I_{max}$, $T/m^2/A$

ℓ length of suspended element,

M magnetization vector, A/m

m_c suspended-element mass, kg

T total torque vector on suspended element, N-m

$\mathbf{T}_c$ magnetic torque vector on suspended element, N-m

$\mathbf{T}_d$ disturbance torque vector on suspended element, N-m

$[\mathbf{T}_E]$ suspended-element rate to Euler rate transformation matrix for a 3, 2, 1 (z, y, x respectively) rotation sequence

$[\mathbf{T}_m]$ inertial coordinate to suspended element coordinate vector transformation matrix

$\mathbf{V}$ velocity vector, m/sec

v permanent magnet core volume, m^3

$\mathbf{W}_1$ weighting matrix (eq. (36))

$\mathbf{W}_2$ modified weighting matrix (eq. (40))

$\mathbf{X}$ state vector for linearized model

x, y, z coordinates in orthogonal axis system, m

δ small increment

θ Euler orientation for 3, 2, 1 rotation sequence, rad

Ω angular velocity vector, rad/sec

Subscripts

b electromagnet axes

m number of coils in system

n coil number

x, y, z component along x, y, z axis respectively

ij partial derivative of i component in j direction

(ij)k partial derivative of ij partial derivative in k direction

max maximum value

o equilibrium condition

Matrix Notation

$[\ ]$ matrix

$[\ ]^{-1}$ inverse of matrix

$\{\ \}$ column vector

$\lfloor\ \rfloor$ row vector

$\lfloor\ \rfloor^T$ transpose of row vector

Dots over a symbol denote derivatives with respect to time; a bar over a symbol indicates that it is referenced to suspended element coordinates.

ANALYTICAL MODEL

This section presents a simplified analytical model of a 6DOF LGMSS which is developed by following the approach detailed in references 2 and 4 using the equations for torques and forces on a cylindrical permanent magnet core uniformly magnetized perpendicular to its axis of symmetry as developed in reference 5. The equations are simplified by using small-angle assumptions and neglecting second-order terms involving suspended-element motion. The permanent magnet core, or suspended element, is levitated over a planar array of electromagnets. Figure 1 is a schematic representation of an eight coil system that shows the coordinate systems and initial alignment. The suspended-element coordinate system consists of a set of orthogonal $\bar{x}$, $\bar{y}$, $\bar{z}$ body-fixed axes that define the motion of the suspended element with respect to an orthogonal x, y, z system fixed in inertial space. The suspended-element coordinate system is initially aligned with the x, y, z system. A set of orthogonal x_b-, y_b-, z_b-axes, also fixed in inertial space, define the location of the electromagnet array with respect to the x, y, z system. The x_b- and y_b-axes are parallel to the x- and y-axes respectively, and the z_b- and z-axes are aligned. The centers of the two axis systems are separated by the distance h. The eight coil array consists of four coils mounted in a circular arrangement in the center and four additional coils mounted around the center array. The array in the center predominantly controls the gradients of the fields and therefore the forces along the x-, y-, and z-axes and the outer array predominantly controls the magnitudes of the fields and therefore the torques about the x-, y-, and z-axes. The cylindrical permanent magnet core, as mentioned above, is magnetized perpendicular to its axis of symmetry and initially the magnetization vector is aligned with the positive z-axis. Therefore, control of the core involves independently controlling the x- and y-components of the field and their gradients in the z direction. As shown in reference 4, independent control of B_x and B_{xz} is not feasible with a single circular array of electromagnets (it can also be shown that independent control of B_y and B_{yz} is not feasible). Hence, two circular arrays with different location radii are employed.

Equations of Motion

From references 2 and 4, the angular acceleration of the suspended element $\dot{\overline{\Omega}}$ in suspended-element coordinates can be written as

$$\dot{\overline{\Omega}} = \left[\mathbf{I_c}\right]^{-1}\overline{\mathbf{T}} \tag{1}$$

where $\left[\mathbf{I_c}\right] = \begin{bmatrix} I_{\bar{x}} & 0 & 0 \\ 0 & I_{\bar{y}} & 0 \\ 0 & 0 & I_{\bar{z}} \end{bmatrix}$ is the moment of inertia about the principal axes of the suspended element and $\overline{\mathbf{T}}$ denotes the total torque on the suspended element. A bar over a variable indicates that it is referenced to suspended-element coordinates. The torque $\overline{\mathbf{T}}$ can be expanded as

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}_c + \overline{\mathbf{T}}_d \tag{2}$$

where $\overline{\mathbf{T}}_c$ denotes the control torque on the suspended element produced by the electromagnets and $\overline{\mathbf{T}}_d$ denotes external disturbance torques. The angular rates of the suspended element are obtained by integrating equation (1). The suspended-element Euler rates can be written as

$$\dot{\theta} = \left[\mathbf{T}_E\right]\overline{\Omega} \tag{3}$$

where $\left[\mathbf{T}_E\right]$ is the suspended-element rate to Euler rate transformation matrix for a 3, 2, 1 (z, y, x) rotation sequence. By using small-angle and rate assumptions, equation (3) reduces to

$$\dot{\theta} \cong \overline{\Omega} \tag{4}$$

where

$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \tag{5}$$

The translational acceleration of the suspended element $\dot{\overline{\mathbf{V}}}$ in suspended-element coordinates can be written as

$$\dot{\overline{\mathbf{V}}} = \left(\frac{1}{m_c}\right) \overline{\mathbf{F}} \tag{6}$$

where m_c is the mass of the suspended element and $\overline{\mathbf{F}}$ denotes the total force on the suspended element. The force $\overline{\mathbf{F}}$ can be expanded as

$$\overline{\mathbf{F}} = \overline{\mathbf{F}}_c + \overline{\mathbf{F}}_d + \overline{\mathbf{F}}_g \tag{7}$$

where $\overline{\mathbf{F}}_c$ denotes control force on the suspended element produced by the electromagnets, $\overline{\mathbf{F}}_d$ denotes external disturbance forces, and $\overline{\mathbf{F}}_g$ consists of the force acting on the suspended element due to gravity, transformed into suspended-element coordinates. The suspended-element translational rates are obtained by integrating equation (6). The suspended-element translational rates $\overline{\mathbf{V}}$ in inertial coordinates are given as

$$\mathbf{V} = \left[\mathbf{T}_m\right]^{-1} \overline{\mathbf{V}} \tag{8}$$

where $\left[\mathbf{T}_m\right]$ is the inertial coordinate to suspended-element coordinate vector-transformation matrix. By using small-angle and rate assumptions, equation (8) reduces to

$$\mathbf{V} \cong \overline{\mathbf{V}} \tag{9}$$

where

$$\mathbf{V} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{10}$$

Magnetic Torques and Forces

From reference 5, the torque on a permanent magnet core which is magnetized perpendicular to the axis of symmetry, in a given coordinate system, can be approximated as

$$T_{cx} = -vM_z B_y \tag{11}$$

$$T_{cy} = vM_z B_x \tag{12}$$

$$T_{cz} = vM_z \left(\frac{\ell^2}{12} - \frac{a^2}{4}\right) B_{(xy)z} \tag{13}$$

where the terms that are a function of second-order gradients have been ignored for T_{cx} and T_{cy}. For simplicity define

$$c_z = \left(\frac{\ell^2}{12} - \frac{a^2}{4}\right) \tag{14}$$

Since $\overline{\mathbf{B}} = \left[\mathbf{T}_m\right]\mathbf{B}$, T_{cx} and T_{cy} in core coordinates can be written as (again using small-angle assumptions)

$$T_{c\bar{x}} = -vM_{\bar{z}}(B_y - \theta_z B_x + \theta_x B_z) \tag{15}$$

$$T_{c\bar{y}} = vM_{\bar{z}}(B_x + \theta_z B_y - \theta_y B_z) \tag{16}$$

Obtaining T_{cz} in core coordinates is more complicated. One method of obtaining $T_{c\bar{z}}$ is to transform the expanded equation for B into core coordinates using equation (A16) in reference 5. The gradients in core coordinates can then be calculated and substituted into equation (40) in reference 5 and the integral over the volume taken. Following this approach, $T_{c\bar{z}}$ becomes

$$T_{c\bar{z}} = vM_{\bar{z}}(c_z B_{(xy)z} + \theta_x c_z (B_{(xz)z} - B_{(xy)y}) + \theta_y c_z (B_{(xx)y} - B_{(yz)z}) + \theta_z c_z (B_{(yy)z} - B_{(xx)z})) \tag{17}$$

From reference 5 the forces on a cylindrical permanent magnet core, for expansion of fields up to second order, are a function of first-order gradients only. From reference 4, the forces in a given coordinate system, as a function of first-order gradients, can be written as

$$\mathbf{F}_c = v[\partial \mathbf{B}]\mathbf{M} \tag{18}$$

where

$$[\partial \mathbf{B}] = \begin{bmatrix} B_{xx} & B_{xy} & B_{xz} \\ B_{xy} & B_{yy} & B_{yz} \\ B_{xz} & B_{yz} & B_{zz} \end{bmatrix} \tag{19}$$

The forces in core coordinates can be written as

$$\overline{\mathbf{F}}_c = v[\mathbf{T}_m][\partial \mathbf{B}][\mathbf{T}_m]^T \overline{\mathbf{M}} \tag{20}$$

For magnetization perpendicular to the axis of symmetry (along the $\bar{z}$ axis) the forces become

$$F_{c\bar{x}} = vM_{\bar{z}}(B_{xz} - \theta_x B_{xy} + \theta_y (B_{xx} - B_{zz}) + \theta_z B_{yz}) \tag{21}$$

$$F_{c\bar{y}} = vM_{\bar{z}}(B_{yz} + \theta_x (B_{zz} - B_{yy}) + \theta_y B_{xy} - \theta_z B_{xz}) \tag{22}$$

$$F_{c\bar{z}} = vM_{\bar{z}}(B_{zz} - 2\theta_x B_{yz} + 2\theta_y B_{xz}) \tag{23}$$

Disturbance Torques and Forces

The assumption is made that the only significant disturbances acting on the suspended element is along the z - axis and is equal to its weight

$$F_g = \begin{bmatrix} 0 \\ 0 \\ -m_c g \end{bmatrix} \tag{24}$$

where g is the acceleration of gravity. Other disturbance torques and forces are ignored. In suspended-element coordinates

$$\overline{F}_g = \left[T_m\right] F_g \tag{25}$$

Performing the transformation (under small-angle assumptions) results in

$$F_{g\bar{x}} = \theta_y m_c g \tag{26}$$

$$F_{g\bar{y}} = -\theta_x m_c g \tag{27}$$

$$F_{g\bar{z}} = -m_c g \tag{28}$$

Linearized Equations

The equations of motion are in the form

$$\ddot{\mathbf{X}} = f\left(\begin{Bmatrix} (\left[\mathbf{I}_c\right]^{-1}\mathbf{T}) \\ ((\frac{1}{m_c})\mathbf{F}) \end{Bmatrix}\right) \tag{29}$$

where

$$\mathbf{X}^{\mathrm{T}} = \lfloor \Omega_{\bar{x}} \;\; \Omega_{\bar{y}} \;\; \Omega_{\bar{z}} \;\; \theta_x \;\; \theta_y \;\; \theta_z \;\; V_{\bar{x}} \;\; V_{\bar{y}} \;\; V_{\bar{z}} \;\; x \;\; y \;\; z \rfloor \tag{30}$$

The torques and forces are functions of **X** and the coil currents **I**; thus,

$$\begin{Bmatrix} ([\mathbf{I_c}]^{-1}\mathbf{T}) \\ ((\frac{1}{m_c})\mathbf{F}) \end{Bmatrix} = f(\mathbf{X},\ \mathbf{I}) \tag{31}$$

where

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} \tag{32}$$

The equations can be linearized around the nominal operating point $\mathbf{X}_o$, $\mathbf{I}_o$ by performing a Taylor series expansion. Neglecting second-order and higher terms and subtracting out $\mathbf{X}_o$ results in

$$\delta\dot{\mathbf{X}} = \mathcal{A}\delta\mathbf{X} + \mathcal{B}\delta\mathbf{I} \tag{33}$$

where

$$\mathcal{A} = \mathbf{W}_1\left(\partial f\left(\begin{Bmatrix} \overline{\mathbf{T}} \\ \overline{\mathbf{F}} \end{Bmatrix}\right) \Big/ \partial\mathbf{X}\right)\Big|_{\mathbf{X}_o,\,\mathbf{I}_o} \tag{34}$$

$$\mathcal{B} = \mathbf{W}_1\left(\partial f\left(\begin{Bmatrix} \overline{\mathbf{T}} \\ \overline{\mathbf{F}} \end{Bmatrix}\right) \Big/ \partial\mathbf{I}\right)\Big|_{\mathbf{X}_o,\,\mathbf{I}_o} \tag{35}$$

and

$$
\mathbf{W}_1 = \begin{bmatrix}
1/I_{\bar{x}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/I_{\bar{y}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/I_{\bar{z}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/m_c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/m_c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/m_c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{36}
$$

Expanding $\mathcal{A}$ results in

$$
\mathcal{A} = \mathbf{W}_1 \begin{bmatrix}
\partial T_{\bar{x}} / \partial \Omega_{\bar{x}} & \partial T_{\bar{x}} / \partial \Omega_{\bar{y}} & \partial T_{\bar{x}} / \partial \Omega_{\bar{z}} & \ldots & \partial T_{\bar{x}} / \partial z \\
\partial T_{\bar{y}} / \partial \Omega_{\bar{x}} & \partial T_{\bar{y}} / \partial \Omega_{\bar{y}} & \ldots & & \\
\partial T_{\bar{z}} / \partial \Omega_{\bar{x}} & \ldots & & & \\
\vdots & & & & \\
\partial V_{\bar{z}} / \partial \Omega_{\bar{x}} & \ldots & & & \partial V_{\bar{z}} / \partial z
\end{bmatrix} \tag{37}
$$

which reduces to

$$
\mathcal{A} = \mathbf{W}_1 \begin{bmatrix}
0 & 0 & 0 & T_{\bar{x}\theta_x} & T_{\bar{x}\theta_y} & T_{\bar{x}\theta_z} & 0 & 0 & 0 & T_{\bar{x}x} & T_{\bar{x}y} & T_{\bar{x}z} \\
0 & 0 & 0 & T_{\bar{y}\theta_x} & T_{\bar{y}\theta_y} & T_{\bar{y}\theta_z} & 0 & 0 & 0 & T_{\bar{y}x} & T_{\bar{y}y} & T_{\bar{y}z} \\
0 & 0 & 0 & T_{\bar{z}\theta_x} & T_{\bar{z}\theta_y} & T_{\bar{z}\theta_z} & 0 & 0 & 0 & T_{\bar{z}x} & T_{\bar{z}y} & T_{\bar{z}z} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F_{\bar{x}\theta_x} & F_{\bar{x}\theta_y} & F_{\bar{x}\theta_z} & 0 & 0 & 0 & F_{\bar{x}x} & F_{\bar{x}y} & F_{\bar{x}z} \\
0 & 0 & 0 & F_{\bar{y}\theta_x} & F_{\bar{y}\theta_y} & F_{\bar{y}\theta_z} & 0 & 0 & 0 & F_{\bar{y}x} & F_{\bar{y}y} & F_{\bar{y}z} \\
0 & 0 & 0 & F_{\bar{z}\theta_x} & F_{\bar{z}\theta_y} & F_{\bar{z}\theta_z} & 0 & 0 & 0 & F_{\bar{z}x} & F_{\bar{z}y} & F_{\bar{z}z} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \tag{38}
$$

Finally, by using the expressions for torques and forces developed earlier (eqns. (15)-(17), (21)-(23), and (26)-(28)), $\mathcal{A}$ becomes

$$
\mathcal{A} = \mathbf{W}_2 \begin{bmatrix}
0 & 0 & 0 & -B_z & 0 & B_x & 0 & 0 & 0 & -B_{xy} & -B_{yy} & -B_{yz} \\
0 & 0 & 0 & 0 & -B_z & B_y & 0 & 0 & 0 & B_{xx} & B_{xy} & B_{xz} \\
0 & 0 & 0 & c_z(B_{(xz)z} - B_{(xy)y}) & c_z(B_{(xx)y} - B_{(yz)z}) & c_z(B_{(yy)z} - B_{(xx)z}) & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -B_{xy} & (\frac{m_c g}{vM_{\bar{z}}} + (B_{xx} - B_{zz})) & B_{yz} & 0 & 0 & 0 & B_{(xx)z} & B_{(xy)z} & B_{(xz)z} \\
0 & 0 & 0 & ((B_{zz} - B_{yy}) - \frac{m_c g}{vM_{\bar{z}}}) & B_{xy} & -B_{xz} & 0 & 0 & 0 & B_{(xy)z} & B_{(yy)z} & B_{(yz)z} \\
0 & 0 & 0 & -2B_{yz} & 2B_{xz} & 0 & 0 & 0 & 0 & B_{(xz)z} & B_{(yz)z} & B_{(zz)z} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \tag{39}
$$

where

$$
\mathbf{W}_2 = \begin{bmatrix}
vM_{\bar z}/I_{\bar x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & vM_{\bar z}/I_{\bar y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & vM_{\bar z}/I_{\bar z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & vM_{\bar z}/m_c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & vM_{\bar z}/m_c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & vM_{\bar z}/m_c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{40}
$$

Next, expanding $\mathcal{B}$ results in

$$
\mathcal{B} = \mathbf{W}_1 \begin{bmatrix}
\partial T_{\bar x}/\partial I_1 & \partial T_{\bar x}/\partial I_2 & \dots & \partial T_{\bar x}/\partial I_m \\
\partial T_{\bar y}/\partial I_1 & \partial T_{\bar y}/\partial I_2 & \dots & \\
\partial T_{\bar z}/\partial I_1 & \dots & & \\
\vdots & & & \\
\partial V_{\bar z}/\partial I_1 & & &
\end{bmatrix} \tag{41}
$$

Evaluating the first term in equation (41) at $\mathbf{X}_o$ results in

$$
\partial T_{\bar x}/\partial I_1\Big|_{\mathbf{X}_o} = -vM_{\bar z}\,\partial B_y/\partial I_1 \tag{42}
$$

Since the fields and gradients are linear functions of coil currents, the components of B_y produced by coil n of an m-coil system can be written as

$$
B_{yn} = k_{yn}(I_n / I_{max}) \tag{43}
$$

where I_{max} is the maximum coil current, k_{yn} is a constant that represents the magnitude of B_{yn} produced by I_{max}, and I_n is the coil current. To simplify, define

$$K_{yn} = k_{yn}/I_{max} \tag{44}$$

For the total system, B_y can be written as

$$B_y = \lfloor \mathbf{K}_y \rfloor \mathbf{I} \tag{45}$$

where

$$K_y = \lfloor K_{y1} \quad K_{y2} \quad \dots \quad K_{ym} \rfloor \tag{46}$$

and **I** is defined by equation (32). Since the elements of $\lfloor \mathbf{K}_y \rfloor$ are constants

$$\partial B_y \,/\, \partial \mathbf{I} = \lfloor \mathbf{K}_y \rfloor \tag{47}$$

Similar results are obtained for the other fields and gradients. Terms in B related to the identities $\dot{\theta}_x = \Omega_{\bar{x}}$, $\dot{\theta}_y = \Omega_{\bar{y}}$, $\dot{\theta}_z = \Omega_{\bar{z}}$, $\dot{x} = V_{\bar{x}}$, $\dot{y} = V_{\bar{y}}$, and $\dot{z} = V_{\bar{z}}$ are zero. Then $\mathcal{B}$ becomes

$$\mathcal{B} = W_2 \begin{bmatrix} -\lfloor K_y \rfloor \\ \lfloor K_x \rfloor \\ c_z \lfloor K_{(xy)z} \rfloor \\ \lfloor 0 \rfloor \\ \lfloor 0 \rfloor \\ \lfloor 0 \rfloor \\ \lfloor K_{xz} \rfloor \\ \lfloor K_{yz} \rfloor \\ \lfloor K_{zz} \rfloor \\ \lfloor 0 \rfloor \\ \lfloor 0 \rfloor \\ \lfloor 0 \rfloor \end{bmatrix} \tag{48}$$

Initial Conditions

The suspended element is assumed to be initially suspended in equilibrium at a distance h above the electromagnet array with the suspended-element coordinates initially aligned with the inertial coordinates as shown in figure 1. In equilibrium, $F_{\bar{x}} = F_{\bar{y}} = 0$ and

$$F_{\bar{z}} = m_c g \tag{49}$$

From equations (15)-(17) and (21)-(23), we have

$$B_x = B_y = B_{(xy)z} = B_{xz} = B_{yz} = 0 \tag{50}$$

and

$$B_{zz} = \frac{m_c g}{v M_{\bar{z}}} \tag{51}$$

In equilibrium, by using the relationship of equation (51), elements (7, 5) and (8,4) of the matrix in equation (39) reduce to

$$\frac{m_c g}{v M_{\bar{z}}} + (B_{xx} - B_{zz}) = B_{xx} \tag{52}$$

and

$$(B_{zz} - B_{yy}) - \frac{m_c g}{v M_{\bar{z}}} = -B_{yy} \tag{53}$$

From equation (45), the controlled fields and gradients as a function of $\mathbf{I}_o$ can be written as

$$\begin{bmatrix} B_x \\ B_y \\ B_{xz} \\ B_{yz} \\ B_{zz} \\ B_{(xy)z} \end{bmatrix} = \begin{bmatrix} \lfloor \mathbf{K}_x \rfloor \\ \lfloor \mathbf{K}_y \rfloor \\ \lfloor \mathbf{K}_{xz} \rfloor \\ \lfloor \mathbf{K}_{yz} \rfloor \\ \lfloor \mathbf{K}_{zz} \rfloor \\ \lfloor \mathbf{K}_{(xy)z} \rfloor \end{bmatrix} \mathbf{I}_{\mathrm{o}} \tag{54}$$

$\mathbf{I}_{\mathrm{o}}$ can be found by inverting the **K** matrix in equation (54) using the generalized inverse. This produces a solution where the 2-norm of the current vector is minimized (i.e., minimum $\sum \mathbf{I}^2$). Once $\mathbf{I}_{\mathrm{o}}$ is determined, the uncontrolled fields and gradients required to complete the $\mathcal{A}$ matrix can be calculated.

As noted in references 3 and 4, one of the objectives of the LGMSS development is to allow positioning of the suspended element through large angles in yaw (θ_z) up to 360^0. As the suspended element is rotated, the equilibrium currents will change. Following the approach detailed in the Appendix of reference 2, the equilibrium currents can be developed as a function of yaw angle and initial torques and forces on the suspended element. Assuming only yaw displacement, $\left[\mathbf{T}_{\mathrm{m}}\right]$ becomes

$$\left[\mathbf{T}_{\mathrm{m}}\right] = \begin{bmatrix} c\theta_z & s\theta_z & 0 \\ -s\theta_z & c\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{55}$$

Since $\overline{\mathbf{B}} = \left[\mathbf{T}_{\mathrm{m}}\right]\mathbf{B}$, from equations (11) and (12)

$$\begin{bmatrix} T_{c\bar{x}} \\ T_{c\bar{y}} \end{bmatrix} = vM_{\bar{z}} \begin{bmatrix} (s\theta_z B_x - c\theta_z B_y) \\ (c\theta_z B_x + s\theta_z B_y) \end{bmatrix} \tag{56}$$

Using the approach discussed earlier to obtain equation (17), $T_{c\bar{z}}$ becomes

$$T_{c\bar{z}} = vM_{\bar{z}} c_z ((c^2\theta_z - s^2\theta_z) B_{(xy)z} + c\theta_z s\theta_z (B_{(yy)z} - B_{(xx)z})) \tag{57}$$

Simplifying equation (57) results in

$$T_{c\bar{z}} = vM_{\bar{z}}c_z(c2\theta_z B_{(xy)z} + \frac{1}{2}s2\theta_z(B_{(yy)z} - B_{(xx)z})) \qquad (58)$$

The forces, from equation (20) become

$$\begin{bmatrix} F_{c\bar{x}} \\ F_{c\bar{y}} \\ F_{c\bar{z}} \end{bmatrix} = vM_{\bar{z}} \begin{bmatrix} (c\theta_z B_{xz} + s\theta_z B_{yz}) \\ (c\theta_z B_{yz} - s\theta_z B_{xz}) \\ B_{zz} \end{bmatrix} \qquad (59)$$

In terms of yaw angle and coil currents, the torques and forces become

$$\begin{bmatrix} T_{c\bar{x}} \\ T_{c\bar{y}} \\ T_{c\bar{z}} \\ F_{c\bar{x}} \\ F_{c\bar{y}} \\ F_{c\bar{z}} \end{bmatrix} = vM_{\bar{z}} \begin{bmatrix} (s\theta_z \lfloor K_x \rfloor - c\theta_z \lfloor K_y \rfloor) \\ (c\theta_z \lfloor K_x \rfloor + s\theta_z \lfloor K_y \rfloor) \\ c_z(c2\theta_z \lfloor K_{(xy)z} \rfloor + \frac{1}{2}s2\theta_z(\lfloor K_{(yy)z} \rfloor - \lfloor K_{(xx)z} \rfloor)) \\ (c\theta_z \lfloor K_{xz} \rfloor + s\theta_z \lfloor K_{yz} \rfloor) \\ (c\theta_z \lfloor K_{yz} \rfloor - s\theta_z \lfloor K_{xz} \rfloor) \\ \lfloor K_{zz} \rfloor \end{bmatrix} \mathbf{I}_o \qquad (60)$$

CONCLUDING REMARKS

This paper has developed a simplified analytical model of a six-degree-of-freedom large-gap magnetic suspension system. The suspended element is a cylindrical permanent magnet that is magnetized perpendicular to its axis of symmetry and the actuators are air-core electromagnets mounted in a planar array. The analytical model is an open-loop representation with electromagnet currents as inputs. The model should be useful in analyses and simulations, in the development of control system approaches, and in evaluations of overall system performance.

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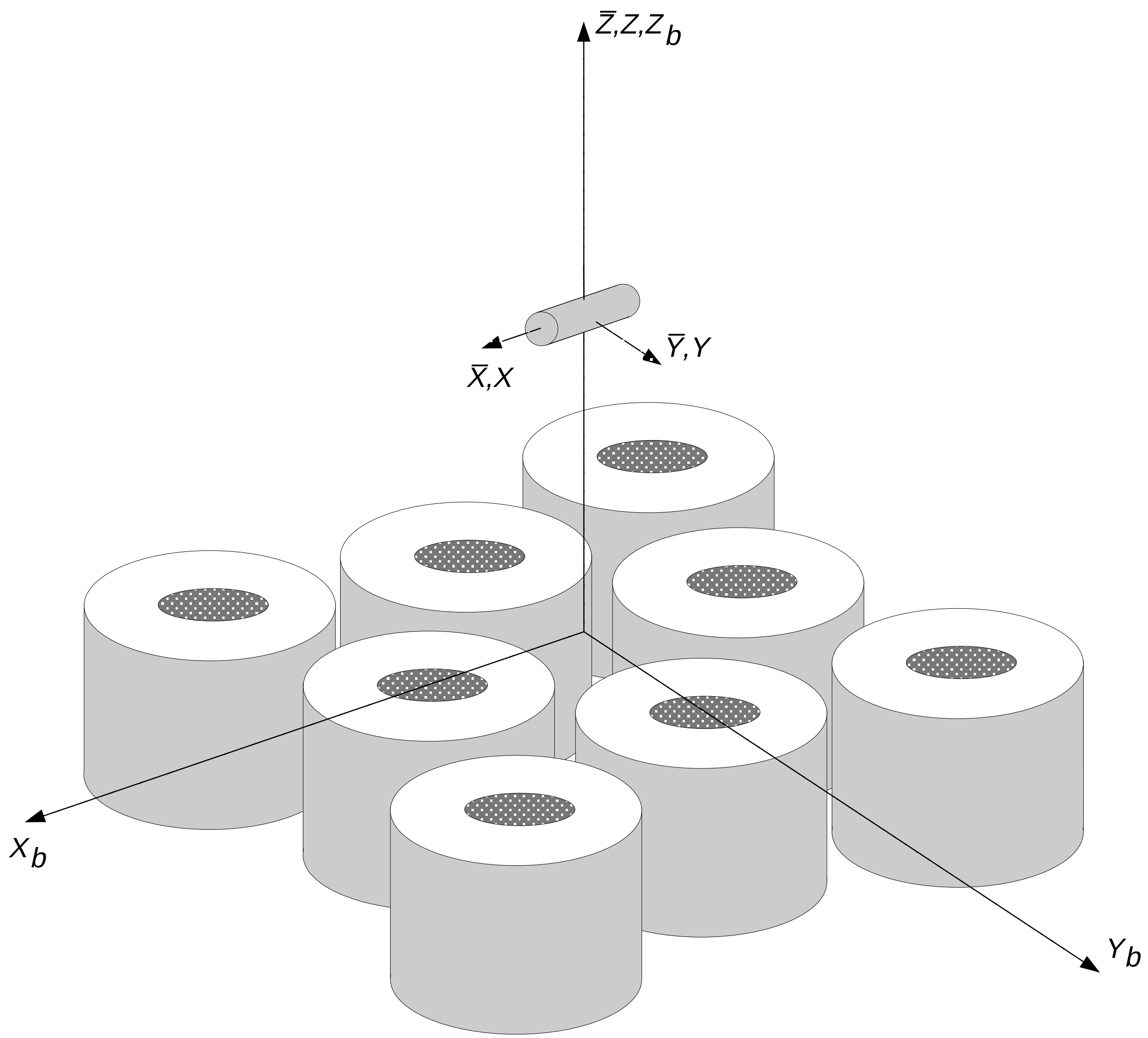

Figure 1. Initial alignment of coordinate systems for 6 DOF Large-Gap Magnetic Suspension System.