Synthesis of Optimal Constant-Gain Positive-Real Controllers for Passive Systems

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Abstract

This paper presents synthesis methods for the design of constant-gain positive real controllers for passive systems. The results presented in this paper, in conjunction with the previous work by the authors on passification of non-passive systems, offer a useful synthesis tool for the design of passivity-based robust controllers for non-passive systems as well. Two synthesis approaches are given for minimizing an LQ-type performance index, resulting in optimal controller gains. Two separate algorithms, one for each of these approaches, are given. The synthesis techniques are demonstrated using two numerical examples: control of a flexible structure and longitudinal control of a fighter aircraft.

Introduction

Passivity-based controllers have been proved to be highly effective in the control of inherently passive systems [Jos.89, Kel.96]. Recently, it has been shown that these controllers can also be used for control of non-passive systems as well once such systems are passified by techniques introduced in [Kel.97, Kel.98]. The main advantage of using such controllers is the stability robustness of the closed-loop system. There are numerous results available in the literature on the passivity-based controllers. However, most of these results have focused on the analysis part and not much work has been done on the synthesis of such controllers. The limited results available to date on the synthesis of positive-real controllers can be found in [Loz.90, Had.94, Saf.87]. In [Loz.90] an LQG-based design technique was given whereas in [Had.94] an H_2/H_∞-based design procedure was given. Both of these methods used stochastic models and are computationally intensive.

This paper gives two approaches to synthesize constant-gain positive-real (PR) controllers. The first approach is based on the use of symmetric gain matrix whereas the second approach allows a non-symmetric gain matrix. In the case of the first approach, the necessary conditions are derived by minimization of a suitable LQ performance index. A synthesis algorithm is derived based on these necessary conditions. For the second approach, a modified version of the algorithm proposed in [Moe.85] is used. The organization of the paper is as follows. First, we present the problem formulation followed by two separate algorithms to compute optimal controller gains, and finally, two numerical examples to demonstrate the synthesis methods.

Positive-Real Optimal Controllers

This section gives the formulation of optimal control problem wherein it is desired to synthesize a constant-gain strictly PR (SPR) output feedback controller for a Linear Time Invariant (LTI) PR system. Consider a Positive-Real (PR) LTI system:

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^m$, and input $u(t)$ is given by the output-feedback control law:

$$u = -Gy \quad (G = G^T > 0). \tag{3}$$

The plant (1) being PR satisfies the following constraints as a result of Kalman-Yakubovich lemma for

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some positive definite $P = P^T \in \Re^{n\times n}$, and $L \in \Re^{m\times n}$.

$$A^T P + PA = -L^T L \quad (4)$$
$$C = B^T P \quad (5)$$

Let us suppose that it is required to find an asymptotically stabilizing constant-gain, strictly positive-real output feedback controller G for the closed-loop system given by Eqs. (1)-(3) such that the following performance index is minimized.

$$J = \int_0^\infty (x^T Q x + u^T R u) dt \quad (6)$$

This is an optimization problem where the performance index (6) is to be minimized subject to the constraints (1)-(5). Using control law of Eq. (3) the closed-loop system becomes

$$\dot{x} = (A - BGC)x. \quad (7)$$

The constraint that G is SPR can be imposed by forcing G to satisfy

$$G = KK^T \quad (8)$$

where $K \in \Re^{m\times m}$ has full rank. Substituting Eq. (8) in Eq. (7) yields

$$\dot{x} = (A - BKK^T C)x = A_{cl}x \quad (9)$$

where $A_{cl} = A - BKK^T C$. The performance function (6) can be rewritten using Eq. (3) as

$$J = \int_0^\infty (x^T (Q + C^T K^T K R K K^T C) x) dt. \quad (10)$$

Further, it can be shown that

$$J = \frac{1}{2} x^T(0) \Sigma x(0). \quad (11)$$

where Σ satisfies:

$$A_{cl}^T \Sigma + \Sigma A_{cl} + Q + C^T K^T K R K K^T C = 0. \quad (12)$$

For a given K, R, and Q this is a Lyapunov equation to be solved for Σ. Now using the trace identity, tr(AB)=tr(BA), the performance index J, can be rewritten as

$$J = \mathrm{tr}(\Sigma X_0) \quad (13)$$

where

$$X_0 \equiv x(0)x^T(0). \quad (14)$$

Assuming $x(0)$ to be a random vector, J in Eq. (13) can be replaced by its expected value, and thus X_0 in Eq. (14) denotes $\mathcal{E}[x(0)x^T(0)]$.

The optimization problem stated above can now be re-phrased as follows:

$$\underset{K,\Sigma}{\text{Minimize}} \quad : \quad J = \mathrm{tr}(\Sigma X_0) \quad (15)$$
$$\text{subject to} \quad : \quad g \equiv {A_{cl}}^T \Sigma + \Sigma A_{cl} + \widetilde{Q} = 0 \quad (16)$$

where $\widetilde{Q} = Q + C^T G^T R G C$. The problem defined by Eqs.(15-16) is a constrained minimization problem which can be further modified into an unconstrained minimization problem by using Lagrange multipliers and augmented performance function.

Let $S = S^T$ be the Lagrange multiplier with consistent dimensions. If Hamiltonian is defined as:

$$\begin{aligned} H &= \mathrm{tr}(\Sigma X_0 + gS) = \mathrm{tr}(\Sigma X_0) \\ &+ tr\Big([A - BKK^T C]^T \Sigma S + \Sigma [A - BKK^T C] S \\ &+ QS + C^T K K^T R K K^T C S\Big) \end{aligned} \quad (17)$$

the constrained optimum of Eq. (15) subject to constraint (16) is same as the unconstrained optimum of Eq. (17). The necessary conditions for an optimum are then given by setting the partial derivatives of H with respect to its arguments to zero, i. e.,

$$\frac{\partial H}{\partial S} = 0 \Rightarrow g = {A_{cl}}^T \Sigma + \Sigma A_{cl} + \widetilde{Q} = 0 \quad (18)$$

$$\begin{aligned} \frac{\partial H}{\partial K} = 0 \Rightarrow & \frac{\partial}{\partial K}\mathrm{tr}\Big(- [C^T K K^T B^T \Sigma S + \Sigma B K K^T C S] \\ & + C^T K K^T R K K^T C S\Big) = 0 \\ \Rightarrow & -[CS^T \Sigma^T B + B^T \Sigma S C^T + B^T \Sigma^T S^T C^T \\ & + CS\Sigma B] K + 2[CSC^T K K^T R \\ & + RKK^T CSC^T] K = 0 \\ \Rightarrow & [CSC^T K K^T R + RKK^T CSC^T - CS\Sigma B \\ & - B^T \Sigma S C^T] K = 0 \end{aligned} \quad (19)$$

$$\frac{\partial H}{\partial \Sigma} = 0 \Rightarrow X_0 + A_{cl} S + S {A_{cl}}^T = 0 \quad (20)$$

Note that the equation (18) is just the constraint equation, whereas Eqs. (19) and (20) give necessary conditions for an optimum. The unconstrained optimum is a local minimum if the Hessian H_{KK} is positive definite.

The following theorem gives conditions under which the the performance function (15) decreases, and leads to a synthesis algorithm.

Theorem 1- *If Σ^1, Σ^0, K^1, and K^0 are solutions to the following equations*

$$A_c{}^{0T}\Sigma^0 + \Sigma^0 A_c{}^0 + \widetilde{Q^0} = 0 \tag{21}$$

$$A_c{}^{1T}\Sigma^1 + \Sigma^1 A_c{}^1 + \widetilde{Q^1} = 0 \tag{22}$$

$$X_0 + A_c{}^0 S^0 + S^0 A_c{}^{0T} = 0 \tag{23}$$

where: $A_c^i = A - BG^iC = A - BK^iK^{iT}C$ and $\widetilde{Q}^i = Q^i + C^TG^{iT}RG^iC = Q^i + C^TK^iK^{iT}RK^iK^{iT}C$ and if K^1 satisfies

$$\begin{aligned} CS^0C^TK^1K^{1T}R \;&+\; RK^1K^{1T}CS^0C^T - \\ CS^0\Sigma^1B \;&-\; B^T\Sigma^1S^0C^T = 0 \end{aligned} \tag{24}$$

then

$$J(K^0) - J(K^1) \geq 0. \tag{25}$$

provided that A_c^i is Hurwitz for $i = 0, 1$.

Proof- *The proof is omitted due to space limitations.*

Using the necessary conditions (Eqs. 19 and 20) and Theorem 1, a numerical algorithm is presented below which can be used to synthesize the PR gain matrix G.

Synthesis of symmetric PR gain: Algorithm 1

Using Theorem 1, the following iterative algorithm can be obtained to compute the gain G.

Step 1. Choose K^0 and solve for Σ^0 using equation (21).

Step 2. Solve for S^0 using equation (23).

Step 3. Solve for K^1 *and* Σ^1 simutanously by using equations (22) and (24).

Step 4. Set $K^0 = K^1$, go back to step 1.

Iterate until convergence is obtained.

It should be noted that Step 3, which consists of solving nonlinear coupled matrix equations, is numerically quite intensive.

Synthesis of non-symmetric PR gain: Algorithm 2

For robust stabilization of passive systems, G does not have to be symmetric, and it suffices to have $G + G^T \geq 0$. This condition is less restrictive than requiring symmetry, and therefore would generally result in a smaller optimal value of the performance function.

In [Moe.85], an algorithm was given for solving the standard LQ output feedback problem. It essentially involves solving two uncoupled Lyapunov equations at each iteration, and then restricting the step size in the direction of the resulting new value of G to ensure closed-loop stability as well as reduction in J. The Lyapunov equations are linear, and therefore the computational requirement is quite reasonable. This algorithm can be modified for the design of PR controllers, by restricting the step size in the direction of new G (at each iteration) to ensure that $G + G^T > 0$. This also ensures closed-loop stability at each iteration. Convergence cannot be guaranteed for this procedure because the positivity constraint can drive the gain to the boundary.

Numerical Examples

Example 1: Flexible Structure Control

The first example consists of a flexible space structure with three lightly-damped elastic modes with frequencies (in rad/sec): 1.095, 2.3, and 2.6. The corresponding open-loop damping ratios are: 0.0023, 0.0011, and 0.0019, respectively. Two actuators with collocated rate sensors are assumed, resulting in a weakly strictly positive real (WSPR) plant. An LQ performance index given by Eq. (6) is minimized by assuming the covariance of the initial state $X_o = I$, and choosing the design variables $Q =$ diag [10, 100, 10, 10, 10, 100], and $R = I_{2\times 2}$.

Results with Algorithm 1:

Algorithm 1 was first used for designing symmetric constant-gain optimal PR controller. The initial gain matrix was chosen to be diag[10, 10]. The initial value of J was 6.1398×10^3. The minimization of J yielded an optimum value of J as 2.5276×10^3. The resulting optimal symmetric, positive gain matrix was

$$G = \begin{bmatrix} 37.5665 & 59.9666 \\ 59.9666 & 214.7910 \end{bmatrix}. \tag{26}$$

The closed-loop eigenvalues of the system are given in Table 1. The position response at both sensor locations are given in Figs. 1 and 2. It can be seen that the closed-loop response dies down within 30 seconds in both cases.

Results with Algorithm 2:

Algorithm 2, which permits non-symmetric gain

matrix G, was next applied to the same problem, starting with the same initial value of G. The algorithm converged in 28 iterations, and the final value of J was: 2.5124e+03, virtually same as that obtained by Algorithm 1. The final optimal value of G was:

$$G = \begin{bmatrix} 31.7064 & 36.7063 \\ 64.3397 & 215.8499 \end{bmatrix}$$

The closed-loop eigenvalues, as well as the initial condition responses, were very close to those obtained by Algorithm 1. When different starting values of G were used, both algorithms converged to nearly the same final values.

The example shows that both algorithms can effectively design an optimal constant-gain positive-real controller.

Example 2: Longitudinal Control System for F-18 Fighter Aircraft

In the second example, linearized longitudinal models of an F-18 High-Alpha Research Vehicle (HARV) at four different flight conditions are considered. The objective is to design a pitch-axis control system at 15,000 ft. altitude, and at the following combinations of speed and normal acceleration: (1) 0.7 Mach and 1g, (2) 0.6 Mach and 1g, (3) 0.49 Mach and 1g, and (4) 0.3 Mach and 0.37g. The control input is elevator deflection and the output is the pitch rate. This is the same system that was considered in [Kel.97], wherein the plant was first passified using a third-order series compensator with poles at -10, -.05, and $-.0035$ and zeros at -1, -0.5, and -0.08. This compensator robustly passifies the plant at all four flight conditions.

The presence of the phugoid mode and the corresponding zero causes numerical problems in control design algorithms. Therefore, passified short-period approximations (of order 5) were used for the purpose of controller design. The LQ performance function to be minimized has $Q = I_{5\times5}$ and $R = 10$. The model corresponding to the second flight condition was used as the nominal model for controller design.

Algorithm 1 was first used for designing a constant-gain PR controller for minimizing the LQ performance index J given by Eq. (6). The initial gain was chosen to be 0.1. The initial value of J was 1.09×10^4. The minimization of J yielded an optimum with value of J as 483.3. The resulting gain was found to be 5.3264. The closed-loop eigenvalues for the nominal plant are given in Table 1. The pitch rate responses for all flight conditions are given in Fig. 3.

Algorithm 2 gave essentially identical results, as is expected because the controller gain is a scalar. The final compensator consists of the passifier in series with the plant and the optimal gain in the feedback path. The responses obtained were notably better than those obtained in [Kel.97] using LQG-optimal dynamic PR controllers of [Loz.90]. This could be attributed to the fact that only limited freedom is available in choosing the performance function weights in the latter case.

Gain Scheduling: The next step was to design optimal controllers tuned to individual flight conditions. In each case, both algorithms converged to essentially same optima. The optimal gains for the four flight conditions are given in Table 2. The responses using individual optimized gains showed a slight improvement over those obtained using the optimal gain for the nominal design model (flight condition 2). If desired, the gains showed in Table 2 can be used for gain scheduling.

Conclusions

Synthesis of constant-gain positive-real LQ-optimal controllers was investigated for passive LTI systems. The controller design technique was demonstrated by two numerical examples. The synthesis methods presented in this paper, along with the robust passification methods proposed in previous publications by the authors, offer an effective tool for designing robust controllers for non-passive systems as well.

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Table 1: Closed-loop eigenvalues

Closed-loop Eigenvalues	
Algorithm 1	*Algorithm 2*
-0.1585 + 1.1218i	± 0.1370 + 1.1170i
-0.6608 + 2.1787i	± 0.7000 + 2.1974i
-0.1627 + 2.5344i	± 0.1587 + 2.5190i

Table 2: Optimal gains

Flight Cond.	1	2	3	4
Optimal gain (G)	5.56	5.33	4.33	4.82

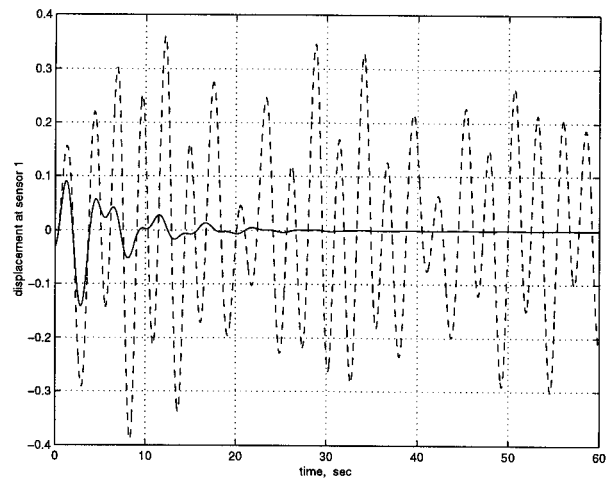

Figure 1: Open-loop (dashed) and closed-loop (solid) response at sensor 1

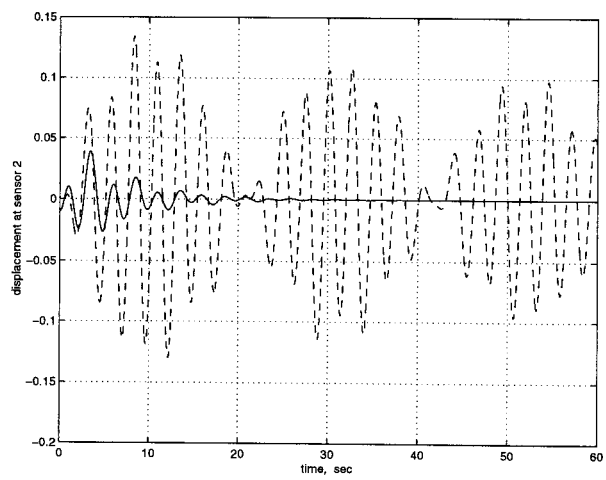

Figure 2: Open-loop (dashed) and closed-loop (solid) response at sensor 2

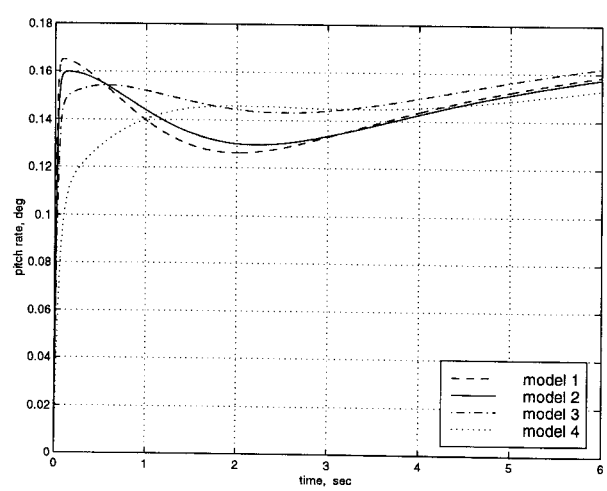

Figure 3: Pitch rate response