

## VOLTERRA SERIES APPROACH FOR NONLINEAR AEROELASTIC RESPONSE OF 2-D LIFTING SURFACES

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### ABSTRACT

The problem of the determination of the subcritical aeroelastic response and flutter instability of nonlinear two-dimensional lifting surfaces in an incompressible flow-field via Volterra series approach is addressed. The related aeroelastic governing equations are based upon the inclusion of structural nonlinearities, of the linear unsteady aerodynamics and consideration of an arbitrary time-dependent external pressure pulse. Unsteady aeroelastic nonlinear kernels are determined, and based on these, frequency and time histories of the subcritical aeroelastic response are obtained, and in this context the influence of geometric nonlinearities is emphasized. Conclusions and results displaying the implications of the considered effects are supplied.

### NOMENCLATURE

$a$  Dimensionless elastic axis position measured from the midchord, positive aft  
 $c$  Chord length of 2-D lifting surface,  $2b$   
 $c_{hi}, c_{\alpha i}, K_{hi}, K_{\alpha i}$  Damping and stiffness coefficients in plunging and pitching ( $i=1,2,3$  - linear, quadratic, cubic), respectively  
 $C_{L\alpha}$  Lift-curve slope  
 $C(k), F(k), G(k)$  Theodorsen's function and its real and imaginary counterparts, respectively  
 $h, \xi$  Plunging displacement and its dimensionless counterpart,  $(h/b)$ , respectively  
 $h_n, H_n$   $n$ -th order Volterra kernel in time, and its Laplace transformed counterpart, respectively

$I_\alpha, r_\alpha$  Mass moment of inertia per unit wingspan and the dimensionless radius of gyration,  $(I_\alpha/mb^2)^{1/2}$ , respectively  
 $l_a, m_a$  Dimensionless aerodynamic lift and moment,  $(L_a b/mU_\infty^2)$  and  $(M_a b^2/I_\alpha U_\infty^2)$ , respectively  
 $L_a, M_a$  Total lift and moment per unit span  
 $L_b, l_b$  Overpressure signature of the N-wave shock pulse and its dimensionless counterpart,  $(L_b b/mU_\infty^2)$ , respectively  
 $m, \mu$  Airfoil mass per unit length and reduced mass ratio,  $(m/\pi\rho b^2)$ , respectively  
 $N$  Load factor,  $1+h''/g$   
 $P_m, \wp_m$  Peak reflected pressure amplitude and its dimensionless counterpart,  $(P_m b/mU_\infty^2)$ , respectively  
 $r$  Shock pulse length factor  
 $s_j, \mathcal{L}$  Laplace transform variable and Laplace operator, respectively,  $s_j = ik_j; i^2 = -1$   
 $S_\alpha, \chi_\alpha$  Static unbalance about the elastic axis and its dimensionless counterpart,  $S_\alpha/mb$ , respectively  
 $t, \tau_0, \tau$  Time variables and dimensionless counterpart,  $(U_\infty t/b)$ , respectively  
 $t_p, \tau_p$  Positive phase duration, measured from the time of the arrival of the pulse, and its dimensionless value, respectively  
 $TF$  Transfer function  
 $U_\infty, V$  Freestream speed and its dimensionless counterpart,  $(U_\infty/b\omega_\alpha)$   
 $x(t)$  Time-dependent external pulse (traveling gusts and wake blast waves)  
 $y(t)$  Response of the considered degree of freedom (pitch  $\alpha$  and/or plunge  $h$ )  
 $\alpha$  Twist angle about the pitch axis  
 $\zeta_h, \zeta_\alpha$  Structural damping ratios in plunging  $(c_h/2m\omega_h)$ , and pitching  $(c_\alpha/2I_\alpha\omega_\alpha)$ ,

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	respectively
$\rho$	Air density
$\phi(\tau), \psi(\tau)$	Wagner and Küssner's functions in the time domain, respectively
$\omega, k$	Circular and reduced frequencies, $(\omega b/U_\infty)$ , respectively
$\omega_h, \omega_\alpha$	Uncoupled frequencies in plunging and pitching, $(K_h/m)^{1/2}$ and $(K_\alpha/I_\alpha)^{1/2}$ , respectively
$\bar{\omega}$	Plunging-pitching frequency ratio, $(\omega_h/\omega_\alpha)$

### Superscript

$(\hat{\cdot})$	Variables in Laplace transformed space
$(\dot{\cdot}), (\cdot)$	Derivatives with respect to the time $t$ , and the dimensionless time $\tau$ , respectively

## INTRODUCTION

It is a well-known fact that within the linearized approach of the aeroelasticity discipline, it is possible to obtain the divergence and the flutter instability boundaries, and also to get the linearized subcritical aeroelastic response of flying vehicle exposed to time-dependent external pulses. On the other hand, in addition to the above mentioned items, the nonlinear approach of the problem can provide important information about the nature of the instability boundary, i.e. benign or catastrophic one, and on the influence of the considered nonlinearities on the subcritical aeroelastic response. In other words, such an approach gives the possibility of determining in what conditions the flutter speed can be exceeded without the occurrence of a catastrophic failure of the lifting surfaces, in which case the flutter is *benign*, as well as the conditions in which undamped oscillations may appear at velocities below the flutter velocity, in which case the flutter is *catastrophic*. Due to the strong implications of various nonlinearities on the highly flexible lifting surfaces, their related aeroelastic phenomena cannot longer be analyzed solely within the standard linearized aeroelasticity theory. Aircraft wing structures often exhibit nonlinearities, which affect their aeroelastic behavior and performance characteristics and flutter boundaries. In order to investigate the aeroelastic behavior of the aircraft in the subcritical flight speed range, and in the vicinity of the flutter boundary, the aeroelastic governing equations have to be considered in nonlinear form.

This investigation concerns the time and frequency formulations of nonlinear two-dimensional lifting surfaces exposed to an incompressible flow field and subjected to an external pressure pulse<sup>1-3</sup>.

Based on Volterra's functional series approach<sup>4-8</sup> important information about the effects of nonlinearities on either the aeroelastic response in the subcritical flight speed regime, and their implication on the benign or catastrophic character of the flutter boundary are supplied.

The advantage of the technique based on Volterra's series and indicial function (Lomax<sup>9</sup>, Bisplinghoff<sup>10</sup> Marzocca et al.<sup>11-12</sup>) consists, among others, on the possibility to investigate, within a rigorous theoretical basis, the aeroelastic systems featuring a wide class of structural nonlinearities.

First of all, based upon the first order Volterra kernel the study of the aeroelastic stability of the systems can be carried out. Moreover, this methodology can encompass the case of an arbitrary number of degrees of freedom and at the same time is conceptually clearer, computationally simpler and can provide more accurate and realistic results as compared to the conventional techniques used in nonlinear aeroelastic systems based on perturbation and multiple scale methods.

Toward the end of determining the nonlinear unsteady aeroelastic kernels, the harmonic probing algorithm, referred to as the method of growing exponentials advanced by Bedrosian and Rice<sup>13</sup>, and the multidimensional Laplace transform will be used.

In addition to the aeroelastic response and determination of the flutter instability boundary, Volterra Series will be used to study the conditions rendering the flutter boundary a benign or a catastrophic one (Librescu<sup>14-15</sup>).

Moreover, when the closed-loop dynamic response of actively controlled lifting surface is analyzed, also the feedback control forces and moments will be included (Librescu<sup>15</sup>, Librescu and Gern<sup>16</sup>, Librescu and Na<sup>17</sup>, Van Trees<sup>18</sup>, Chua and Ng<sup>19</sup>).

The Volterra's series approach provides a firm basis of nonlinear subcritical aeroelastic response, in the sense that it supplies an explicit relationship between the input (any kind of time-dependent external pulses, i.e. blast load, sonic-boom, gust loads) and its response.

With the so-called *Volterra Kernel* identification scheme the modeling of an aeroelastic system using this approach becomes feasible. However, this methodology requires determination for each specific flight conditions of the appropriate nonlinear kernel of the Volterra's series. For this reason, in order to define the appropriate aerodynamic loads, the recent interest in the modeling of unsteady nonlinear aerodynamics by this approach has been focused on the identification of Volterra's kernels in the *time domain* (Silva<sup>20-23</sup>), and in the *frequency domain* (Marzocca et al.<sup>12</sup>, Tromp and Jenkins<sup>24</sup>).

A number of fundamental contributions related with Volterra's series, developed by outstanding mathematicians (Volterra<sup>4</sup>, Wiener<sup>5</sup>) and used mainly in electrical engineering<sup>6-8</sup>, are already available.

The original studies on functional series by Volterra<sup>4</sup>, have been continued in the works by Volterra himself, of those of famous physicists and mathematicians as Rugh<sup>6</sup>, Schetzen<sup>7</sup>, Boyd<sup>8</sup>. These concepts have been used in nonlinear system theory, in general, and in the modeling of nonlinear aeroelastic systems, in particular (Silva<sup>20</sup>). Very few applications of this method have been done in the aeroelasticity discipline.

Originally, the method of Volterra series and Volterra kernel identification were developed to identify the nonlinear behavior in electrical circuits. In the aerospace field, the fundamental contributions were brought by Silva, who has shown that the method is also applicable to aeroelastic systems (aerodynamic reactions and forced structural model).

Silva's pioneering work<sup>20-23</sup>, in this area has opened a very promising way of modeling and approaching nonlinear aeroelastic systems.

### BASIC CONCEPTS AND LIMITATIONS OF THIS APPROACH

Having in view the fact that for nonlinear systems the superposition principle is not applicable, and having in view the different types of responses induced by unsteady aerodynamic loads and the external excitation, a combination of transfer functions is used. These transfer functions for the nonlinear aeroelastic systems and the time-histories response in time and frequency domains are determined by taking the multi-dimensional Laplace transform of the Volterra kernels of the related aeroelastic system via a Mathematica<sup>®</sup> code developed by these authors<sup>25</sup>.

Our approach intended to address the subcritical response of the nonlinear aeroelastic governing equations, is based on its exact representation as an infinite sum of multidimensional convolution integrals, the first one, (i.e. the linear kernel) being the analogous to the linear indicial aeroelastic function. The full nonlinear aeroelastic response will be composed of additional higher-order contributions. In the frequency domain, if the nonlinear function governing a system is 'smooth', then for small inputs the system must be asymptotically linear<sup>6</sup>. One of the key issues is to determine, corresponding to the considered type of structural and aerodynamic nonlinearities, the pertinent Volterra's kernels. When also the active control is implemented the corresponding Volterra's kernel should also be derived.

### THE THEORY

In an attempt to make the paper as self-contained as possible, several elements associated with Volterra's series as applied to aeroelastic system, as well as with the indicial functions will be supplied here.

### Indicial Theory and Aerodynamic Loads

Using the *aerodynamic indicial functions* corresponding to transient aerodynamic reaction to a step pulse, the aerodynamic forces and moments induced in any maneuver and any flight regime can be determined. Aerodynamic forces and moments acting on a rapidly maneuvering aircraft are, in general, nonlinear functions of the motion variables, their time rate of change, and the history of the maneuvering (Tobak & Chapman<sup>26</sup>). However, in this study, the linear aerodynamic theory is adopted.

Once the response of the system to a step change in one of the disturbing variables (i.e. the indicial response) is known, the indicial method permits the determination of the response of a system to an arbitrary schedule of disturbances. There is a critical value of the flight speed above which the steady motion becomes unstable. In a nonlinear aeroelastic system the flutter phenomenon corresponds to the instability known as the Hopf bifurcation, resulting in finite amplitude oscillations, in the case of *supercritical Hopf bifurcation*, and in oscillations with increasing amplitudes, even if the system operates before reaching the flutter speed, in the case of the *subcritical Hopf bifurcation*<sup>27-29</sup>.

We need to mention that a *nonlinear indicial theory*<sup>30</sup>, asserts that the response of a nonlinear system to an arbitrary input can be constructed by integrating a nonlinear functional, that involves the knowledge of the time-dependent input and the kernel response. Whereas, within the linear indicial theory the linear kernel or linear impulse response can be convolved with the input to predict the output of a linear system, the nonlinear indicial theory constitutes a generalization of this concept. It can also be shown that the traditional Volterra-Wiener theory of nonlinear systems constitutes a subset of nonlinear indicial theory. It should also be mentioned that the nonlinear unsteady aerodynamics valid throughout the subsonic incompressible/compressible, transonic and supersonic flight speed regimes can be used and determined via the use of nonlinear indicial functions<sup>31</sup> in conjunction with the Volterra's series approach.

### Volterra Functional Series Theory

As it was shown (Rugh<sup>6</sup>, Schetzen<sup>7</sup>) within Volterra's series approach the full response in the time domain,  $y(t)$ , of the nonlinear systems with memory can be cast as:

$$y(t) = \sum_{k=0}^{\infty} y_k(t), \quad (1)$$

where,  $y_k(t)$  is expressed as:

$$y_k(t) = \int \int \int \dots \int_{-\infty}^{\infty} h_k(t - \tau_1, t - \tau_2, \dots, t - \tau_k) \prod_{i=1}^k x(\tau_i) d\tau_i \quad (2)$$

By a change of variables, it is possible to express Eq. (2) in contracted form as:

$$y_k(t) = \iiint\limits_k \int\limits_{\text{times}} \int\limits_0^\infty h_k(\tau_1, \tau_2, \dots, \tau_k) \prod_{i=1}^k x(t - \tau_i) d\tau_i \quad (3)$$

It is assumed that  $x(t) = 0$  for  $\tau < 0$  implying that the system is causal.

With this restriction, all the integrals in the subsequent discussions are different from zero over the time range  $(0, \infty)$ . Restricting the development of Eq. (3) to the first three terms one obtains:

$$\begin{aligned} y(t) = & \int h_1(\tau_1) x(t - \tau_1) d\tau_1 \\ & + \iint h_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) d\tau_1 d\tau_2 \\ & + \iiint h_3(\tau_1, \tau_2, \tau_3) x(t - \tau_1) x(t - \tau_2) x(t - \tau_3) d\tau_1 d\tau_2 d\tau_3 \\ & + \dots \end{aligned} \quad (4)$$

On the other hand, the response of the system can be expressed also in the frequency domain.

The Volterra series is essentially a polynomial approximation of the system, extension of Taylor series to systems with memory, while Volterra's kernels  $h_i(s_i)$  are a direct extension of the impulse response concept of the linear system theory to multiple dimensions (Volterra<sup>5</sup>, Rugh<sup>6</sup>, Schetzen<sup>7</sup>, Boyd<sup>8</sup>). Consequently, a multidimensional analogue of the impulse response can be used to characterize a nonlinear system (Silva<sup>20-23</sup>).

Having in view that the aeroelastic systems memory is not infinite and, at the same time, the time-dependent external excitations, such as impulse, gust, blast and sonic-boom pressure signatures are non persistent but their effect will diminish as time unfolds, it is possible to characterize a nonlinear aeroelastic system via Volterra series. This fact is reflected in the interpretation of the Volterra kernels as higher order impulse response functions, i.e.  $h(\tau_1, \dots, \tau_n) \rightarrow 0$  as  $\tau_1, \dots, \tau_n \rightarrow \infty$ .

We will use the definition of the *nonlinear transfer function or higher-order impulse response functions* namely:

$$H_k(s_1, s_2, \dots, s_n) = \iint \dots \int h_k(\tau_1, \tau_2, \dots, \tau_k) e^{-s_1\tau_1} e^{-s_2\tau_2} \dots e^{-s_n\tau_n} d\tau_1 d\tau_2 \dots d\tau_k, \quad (5)$$

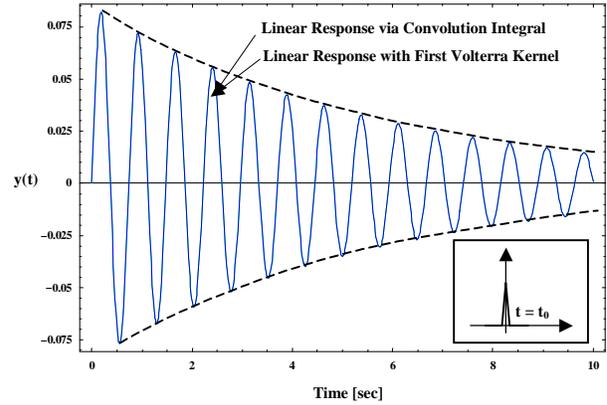
as well as of its inverted counterpart:

$$h_n(\tau_1, \tau_2, \dots, \tau_n) = \left( \frac{1}{2\pi i} \right)^n \int_{\sigma_n - j\infty}^{\sigma_n + j\infty} \dots \int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} H_n(s_1, s_2, \dots, s_n) e^{s_1\tau_1} e^{s_2\tau_2} \dots e^{s_n\tau_n} ds_1 ds_2 \dots ds_n. \quad (6)$$

Once the Volterra's kernels are known the response of the nonlinear aeroelastic system can fully be identified. As demonstrate in the Schetzen works<sup>7</sup>, without loss of generality, the kernels will be taken as symmetric.

If we focalize the attention on the linear system, the Laplace transform  $\mathcal{L}$  of the first term of Eq. (4) yields the familiar Laplace domain expression  $Y(s) = H(s)X(s)$

where  $Y(s), H(s), X(s)$  are the Laplace transforms of  $y(\tau), h(\tau), x(\tau)$ , respectively, and  $H(s)$  is the transfer function of the system; either the first transfer function or the first kernel in time  $h(\tau)$  encode all the information about the aeroelastic system, that is, of course, exact only for the linear system. Moreover, as is well known, if the system is linear, i.e. superposition principle holds valid, and is time invariant, the external load is uniquely related to the response by a convolution integral. With the use of functional series, i.e. the Volterra series, this functional representation can be extended to nonlinear systems. The comparison between the prediction of the linear aeroelastic responses of 2-D lifting surface in incompressible flow field based on the Volterra's series approach (using Theodorsen's function) and on the exact solution, based on convolution integrals (using Wagner's function) is presented in Fig. 1.



**Fig. 1** Aeroelastic response time-history of lifting surface in incompressible flight speed regime to Dirac delta impulse, as represented in inset. Comparison of prediction of responses based on the first Volterra kernel and the exact solution.

The excellent agreement of these two predictions shown here, assess both the accuracy of the aeroelastic model and also the power of the methodology that combines Volterra's series and indicial function.

## MATHEMATICAL FORMULATION

### General theory for 2-D lifting surfaces including structural nonlinearities

The aeroelastic governing equation of motion for 1 and 2 DOF including structural nonlinearities that include the damping and the stiffnesses can be analyzed in the following way. Two systems will be analyzed here: a 1 DOF lifting surface (i.e. plunging only) and a 2 DOF lifting surface featuring structural and aerodynamic coupling in plunging  $h$  and pitching  $\alpha$ . As

previously mentioned, the unsteady aerodynamic is considered linear. A harmonic time dependent external concentrated load is also applied. This configuration for example, can be considered to correspond to an engine mounted on an aircraft wing. As a result, a harmonic type loading due to the engine oscillations has to impact the motion of the wing.

As a characteristic of this approach, the *transfer functions* of the system would exist and be the same for any excitations<sup>32-33</sup>, (namely for random, sine, impulse). This is due to the fact that transfer functions are a characteristic of the system itself and are independent of the input to the system.

As a reminder, the validity of this method is based on the use of continuous polynomial type nonlinearities.

For nonlinear ordinary differential systems, there are in general, an infinite number of Volterra kernels. In practice, one can handle only a finite number of terms in the series, which leads to the problem of truncation accuracy. However, Wiener suggest that the first terms of the series may be sufficient to represent the output of a nonlinear system if the nonlinearities are not too strong.

The use of the multidimensional Laplace transform as a function of several variables is a tool useful in stationary nonlinear system theory. The multivariable convolutions can be represented in terms of products of Laplace transforms.

It is well known that the nonlinear aeroelastic systems cannot be described by a simple transfer function for two main reasons: a) the response has different trends as compared to the unsteady aerodynamic loads and the external excitation and, b) in the nonlinear case the superposition principle is not applicable. It is also well known that any time-dependent external excitation, i.e. periodic or otherwise, can be represented, to an arbitrary degree of accuracy, by a sum of sinusoidal waves<sup>31</sup>. In this context, if the external load is expressed in term of multiple sinusoidal form (for example traveling gust loads) this is easily convertible in the exponential form, i.e.:

$$u(t) = A \cos(\omega_A t) + B \cos(\omega_B t)$$

$$\Leftrightarrow u(t) = \frac{A}{2} (e^{s_A t} + e^{-s_A t}) + \frac{B}{2} (e^{s_B t} + e^{-s_B t}). \quad (7)$$

For clarity of exposition, it is convenient to adopt this approach for a system with one degree of freedom (*1 DOF*). These results have more general bearing and can be extended for systems with multi-degree of freedom (*MDOF*). In fact, by using the classical approach of the one dimensional frequency response function, it is possible to derive an analytical form of the multi - dimensional frequency response characteristics of nonlinear systems. The systems based on 1 DOF (plunging  $h$ ) and 2 DOF, pitching  $\alpha$  and plunging  $h$ , will be considered in the next sections.

### **Pure Plunging Airfoil**

The nonlinear equation of motion of an airfoil featuring plunging motion can be expressed as:

$$m\ddot{h}(t) + \sum_{j=1}^n (c_{hj} \dot{h}(t)^j + k_{hj} h(t)^j) - L_a(t) = L_b(t). \quad (8)$$

where  $k_{hj}$ ,  $c_{hj}$ ,  $m$  are the stiffness, damping and mass parameters, respectively.

In Eq. (8) the related unsteady aerodynamic lift is represented as a function of the plunging degree of freedom  $h$ , only:

$$L_a(\tau) = -C_{L\alpha} \rho U_\infty^2 \int_{-\infty}^{\tau} \phi(\tau - \tau_0) h'' d\tau_0 - \frac{1}{2} \rho C_{L\alpha} U_\infty^2 h'' \cdot \quad (9)$$

The non-circulatory component present in Eq. (9) has been represented in term of convolution integral of the indicial Wagner's function.

In order to explain how this methodology works, let us to determine, in terms of Volterra series, how a system responds to a harmonic or periodic time-dependent load. Let consider a periodic external excitation in the form:

$$L_b(t) = \sum_{j=1}^n X_j e^{s_j t}. \quad (10)$$

As is well known, the information acquired by the case of the response to a harmonically time-dependent load can be used to obtain the response to any time-dependent excitation. In fact, considering the case of a concentrated load arbitrarily located in the  $x, y$  plane of the wing, we have:

$$u(x, y, t) = A \delta(x - x_0, y - y_0) e^{i\omega t}, \quad (11)$$

where  $\delta(\cdot)$ ,  $x_0$ ,  $y_0$ ,  $A$ ,  $\omega$  denote Dirac's distribution, location of the load, its amplitude and excitation frequency, respectively. Once determined the transfer function (labeled as *TF*) corresponding to a given excitation frequency, its counterpart in the time domain can be obtained as the inverse Laplace transform  $\mathcal{L}^{-1}$ :

$$TF(x, y, t) = \mathcal{L}^{-1}\{TF(x, y, s)\} = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} TF(x, y, s) e^{st} ds$$

In addition to the direct role in the determining of the response, the transfer function *TF* has then the role in determining the response to arbitrary time-dependent external excitations.

The general procedure to identify the aeroelastic kernels of various order ( $\overline{1, n}$ ), is to consider a general input in the form of Eq. (10) and to equate coefficients of  $X_1 X_2 \dots X_n e^{(s_1 + s_2 + \dots + s_n)t}$ . As an example, the first aeroelastic Volterra's kernels that describes the linear system in the aeroelastic governing equations, obtained by neglecting the nonlinear terms, is obtained by considering the input load as  $L_b(t) = X_1 e^{s_1 t}$  (which in dimensionless form is expressed as  $l_b(t) = (b/mU_\infty^2) X_1 e^{s_1 t}$ ); the response of the system is postulated in the form  $h(t) = H_1(s_1) X_1 e^{s_1 t} + h.o.t.$ . Substituting  $h(t)$  and its

derivatives in the governing equation of motion, one determines the coefficient of  $X_1 e^{s_1 t}$ .

In a linear aeroelastic system, the system is completely characterized by a transfer function  $H_1(s_1)$  that contains the aerodynamic term as follow:

$$H_1(s_1) = \frac{(k_{h1} + ms_1^2 + c_{h1}s_1 + s_1 \rho C_{L\alpha} b U_\infty C[-is_1 b/U_\infty])}{\frac{1}{2} \rho C_{L\alpha} s_1^2 b^2} \quad (12)$$

Herein the Theodorsen's function  $C$ , connected with the Wagner's indicial function  $\phi(\tau)$  via the Laplace's transform as  $C(-is) = s \int_0^\infty \phi(\tau) e^{-s\tau} d\tau$ , has been included in the formulation. The terms underscored by the solid line correspond to the unsteady aerodynamic loads component (circulatory term), while the dotted line identifies the terms corresponding to the effect of the added mass. When the aerodynamic loads are neglected and for  $s = i\omega$ , this result coincides with that of the linear FRF, derived via the conventional Modal Analysis.

For purely mechanical systems, in the frequency domain, the response analyses via Volterra series have been carried out by several authors. In the present study an alternative procedure, based on the multivariable kernel transforms referred to as Higher-order Transfer Functions (HTFs) is pursued. The two above mentioned approaches can be correlated each other, and this is shown also in this work. Assuming zero initial condition, the frequency response functions (FRFs) are obtained from the transfer functions TFs, by replacing the Laplace transform variable  $s$  with  $j\omega$  where  $\omega$  is the frequency of the excitation, (Worden et al.<sup>33</sup>).

In the present nonlinear aeroelastic system, toward the estimation of higher order frequency response functions (HO FRFs) that are defined as the multi-dimensional Fourier Transform (MDFT) of the Volterra's kernels, a sequence of transfer functions are employed. The concept of higher order of FRFs, independent of the input to the system, defined from the Volterra series, will also be included.

By the use of the linear frequency-response-function  $H_1(s_1)$  the behavior of the linear system is easily determined. It will be necessary to find a complete set of Volterra kernel transforms  $H_n(s_1, s_2, \dots, s_n)$  for nonlinear systems and for this, in practice, we will use a convergent truncated series.

However, probing the system with a single harmonic yields only the information about the value of the transfer functions terms on the diagonal line of the plane  $s_1, s_2$ , in the Laplace transformed space, where  $s_1 = s_2$ . However, in order to obtain information elsewhere in this space, one should use multi-frequency excitations.

In the same way, the second order Volterra Kernel can be determined applying a load depending on two

different frequencies expressed as:  $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t}$ .

In this case we can express the plunging response in the form:

$$\begin{aligned} h(t) = & H_1(s_1) X_1 e^{s_1 t} + H_1(s_2) X_2 e^{s_2 t} \\ & + H_2(s_1, s_1) X_1^2 e^{2s_1 t} + H_2(s_2, s_2) X_2^2 e^{2s_2 t} \\ & + H_2(s_1, s_2) X_1 X_2 e^{(s_1+s_2)t} + H_2(s_2, s_1) X_2 X_1 e^{(s_2+s_1)t} \\ & + h.o.t. \end{aligned} \quad (13)$$

Substituting Eq. (13) in Eq. (8) and equating the terms containing  $X_1 X_2 e^{(s_1+s_2)t}$  the second order aeroelastic Volterra Kernel in the Laplace transformed space is obtained:

$$H_2(s_1, s_2) = -(s_1 s_2 c_{h2} + k_{h2}) H_1(s_1) H_1(s_2) H_1(s_1 + s_2), \quad (14)$$

where:

$$\begin{aligned} H_1(s_1 + s_2) = & (k_{h1} + (s_1 + s_2)^2 m + c_{h1}(s_1 + s_2) \\ & + (s_1 + s_2) \rho C_{L\alpha} b U_\infty C[-i(s_1 + s_2) b/U_\infty] \\ & + \frac{1}{2} \rho C_{L\alpha} (s_1 + s_2)^2 b^2)^{-1} \end{aligned} \quad (15)$$

is the first order Volterra Kernel in the Laplace transformed space at the frequency  $\omega_1 + \omega_2$ . Following the same steps, applying the load  $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t} + X_3 e^{s_3 t}$ , equating the terms in the form  $X_1 X_2 X_3 e^{(s_1+s_2+s_3)t}$ ; remembering that

$$\begin{aligned} H_1(s_1 + s_2) = & (k_1 + (s_1 + s_2)^2 m + c_1(s_1 + s_2) \\ & + (s_1 + s_2) \rho C_{L\alpha} b U_\infty C[-i(s_1 + s_2) b/U_\infty] \\ & + \frac{1}{2} \rho b^2 C_{L\alpha} (s_1 + s_2)^2)^{-1} \end{aligned} \quad (16)$$

$$\begin{aligned} H_1(s_1 + s_2 + s_3) = & (k_1 + (s_1 + s_2 + s_3)^2 m + c_1(s_1 + s_2 + s_3) \\ & + (s_1 + s_2 + s_3) \rho C_{L\alpha} b U_\infty C[-i(s_1 + s_2 + s_3) b/U_\infty] \\ & + \frac{1}{2} \rho b^2 C_{L\alpha} (s_1 + s_2 + s_3)^2)^{-1} \end{aligned} \quad (17)$$

the expressions for the third order Volterra Kernel in the Laplace transformed space can be cast as:

$$\begin{aligned} H_3(s_1, s_2, s_3) = & -\frac{2}{3} (H_1(s_3) (3H_1(s_1) H_1(s_2) (k_{h3} + c_{h3} s_1 s_2 s_3) \\ & + 2H_2(s_1, s_2) (k_{h2} + c_{h2} (s_1 + s_2) s_3)) \\ & + 2(H_1(s_2) H_2(s_3, s_1) (k_{h2} + c_{h2} s_2 (s_1 + s_3)) \\ & - H_1(s_2) H_2(s_2, s_3) (k_{h2} + c_{h2} s_1 (s_2 + s_3)))) / \\ & (H_1(s_1 + s_2 + s_3)) \end{aligned} \quad (18)$$

Notice that the constants  $k_{h2}$  and  $c_{h2}$  multiply the whole expression for  $H_2$ , and this term vanishes if the quadratic term is absent in the aeroelastic governing equation of motion. As one of the general properties of Volterra's series, if all nonlinear terms in the equation of motion for the system are odd powers of  $x$  and  $y$ , then the associated Volterra series have no even-order kernels. As a consequence it will possess no even-order TFs. It is also a general property of systems that all higher-order TFs can be expressed in terms of  $H_1$ . The expressions are function of the system and can be obtained using the harmonic probing algorithm.

### Plunging-Pitching Airfoil

The governing aeroelastic system of an airfoil featuring plunging - twisting coupled motion, exposed to a harmonic time dependent external excitation is:

$$m\dot{h} + S_\alpha \dot{\alpha} + \sum_{j=1}^n (c_{hj} \dot{h}^j + k_{hj} h^j) - L_a = L_b, \quad (19)$$

$$S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + \sum_{j=1}^n (c_{\alpha j} \dot{\alpha}^j + k_{\alpha j} \alpha^j) - M_a = 0. \quad (20)$$

Following the steps adopted for 1 DOF, applying a load depending on one frequency  $L_b = X_1 e^{s_1 t}$ , and expressing the plunging and pitching displacements in terms of transfer functions as:

$$h(t) = X_1 H_1^h(s_1) e^{s_1 t} + X_1^2 H_2^h(s_1, s_1) e^{2s_1 t} + X_1^3 H_3^h(s_1, s_1, s_1) e^{3s_1 t}, \quad (21)$$

$\alpha(t) = X_1 H_1^\alpha(s_1) e^{s_1 t} + X_1^2 H_2^\alpha(s_1, s_1) e^{2s_1 t} + X_1^3 H_3^\alpha(s_1, s_1, s_1) e^{3s_1 t}$ , (22) the relative kernels and the aeroelastic responses can be determined.

The aeroelastic governing system including the blast pressure signatures can be expressed in the Laplace transformed space as:

$$s^2 \hat{\xi} + \chi_\alpha s^2 \hat{\alpha} + 2\zeta_\alpha \frac{\bar{\omega}}{V} s \hat{\xi} + \left(\frac{\bar{\omega}}{V}\right)^2 \hat{\xi} + \frac{2}{\mu} \left( s \hat{\alpha} + s^2 \hat{\xi} + s^2 \left(\frac{1}{2} - a\right) \hat{\alpha} \right) \Phi(s) + \frac{1}{\mu} s^2 (\hat{\xi} - a \hat{\alpha}) + \frac{1}{\mu} s \hat{\alpha} = l_b(s), \quad (23)$$

$$\left(\chi_\alpha / r_\alpha^2\right) s^2 \hat{\xi} + s^2 \hat{\alpha} + (2\zeta_\alpha / V) s \hat{\xi} + \hat{\alpha} / V^2 - \left(\frac{1}{2} + a\right) \frac{2}{\mu} \frac{1}{r_\alpha^2} \left( s \hat{\alpha} + s^2 \hat{\xi} + s^2 \left(\frac{1}{2} - a\right) \hat{\alpha} \right) \Phi(s) - \frac{1}{r_\alpha^2} \frac{1}{\mu} a s^2 (\hat{\xi} - a \hat{\alpha}) + \left(\frac{1}{2} - a\right) \frac{1}{r_\alpha^2} \frac{1}{\mu} s \hat{\alpha} + \frac{1}{8} \frac{1}{r_\alpha^2} \frac{1}{\mu} s^2 \hat{\alpha} = 0. \quad (24)$$

Herein  $\hat{(\cdot)} = \mathcal{L}(\cdot)$ , so  $\hat{\xi} = \mathcal{L}(\xi(t))$  and  $\hat{\alpha} = \mathcal{L}(\alpha(t))$ . Following the same steps, applying the loads  $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t}$  and  $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t} + X_3 e^{s_3 t}$ , equating the terms in the forms  $X_1 X_2 e^{(s_1+s_2)t}$  and  $X_1 X_2 X_3 e^{(s_1+s_2+s_3)t}$  the expressions for the second and third order Volterra Kernel in the Laplace transformed space can be obtained.

### Generalization to Multi Degrees of Freedoms Systems (M-DOFs)

The method shown for 1-DOF and 2-DOF lifting surface can be extended to systems featuring multi degrees of freedoms, in general, and to a 3-D aircraft wing, in particular.

The method of deriving the  $n$ -th order nonlinear aeroelastic transfer functions is based upon the fact that when the aeroelastic system described by the response  $y(t)$  (expressed via Volterra series), is excited by a set of  $k$  unit amplitude exponentials at the arbitrary frequencies  $s_1, s_2, \dots, s_k$ , the output will contain exponential components of the form:

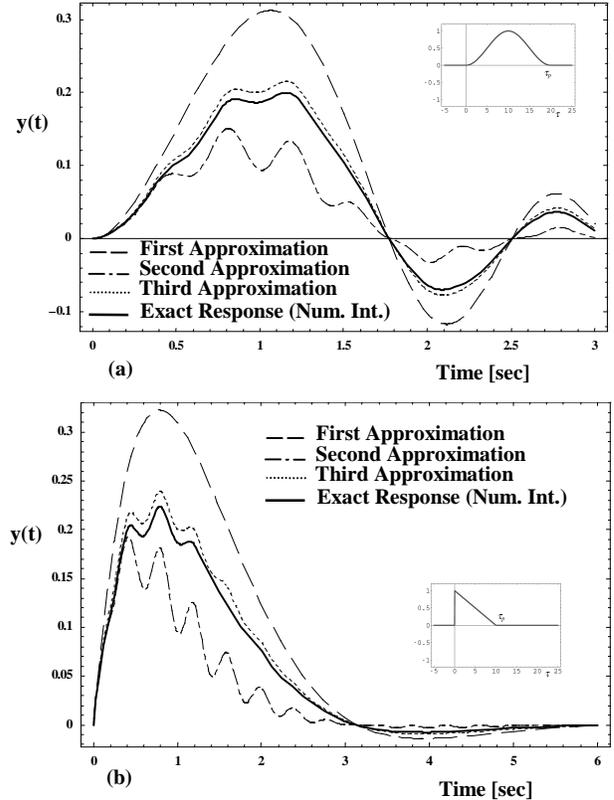
$$y(t) = \sum_{n=1}^{\infty} \sum_{m=1}^k H_n(s_{k1}, s_{k2}, \dots, s_{kn}) e^{(s_{k1}+s_{k2}+\dots+s_{kn})t}. \quad (25)$$

The presence of nonlinearities causes harmonic excitations and sums of harmonics to appear in the response of the aeroelastic system. Due to the nonlinear formulation, different frequencies can be expected as well.

From the energetic point of view, we can observe that  $H_1(s_1)$  produces a single frequency output in response to the simple input  $e^{s_1 t}$ . However, because the system is nonlinear,  $H_2(s_1, s_2)$  takes into account the terms that produce an output energy corresponding to the sum of frequencies  $\omega_1 + \omega_2$ , or in other words to the input  $e^{(s_1+s_2)t}$ . Similarly, the third order nonlinear aeroelastic kernel, will inject a mix of three input frequencies into the total system output.

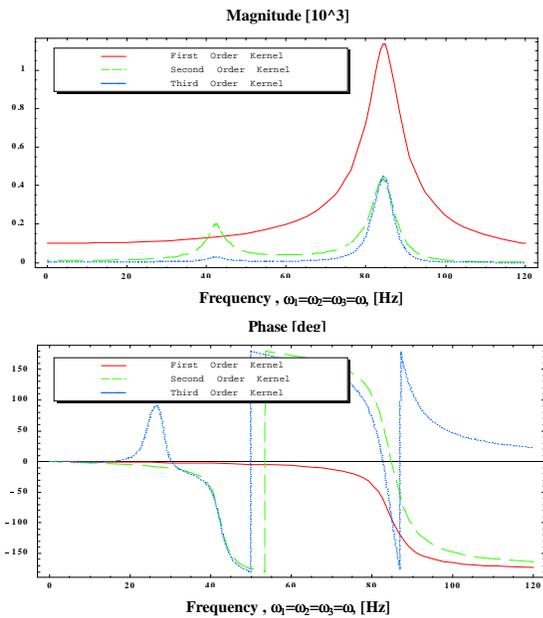
## RESULTS AND DISCUSSIONS

To assess the versatility and provide a validation of this methodology, a comparison of the predictions of the aeroelastic response of nonlinear 2-D lifting surface using three approximations are shown in Figs. 2.



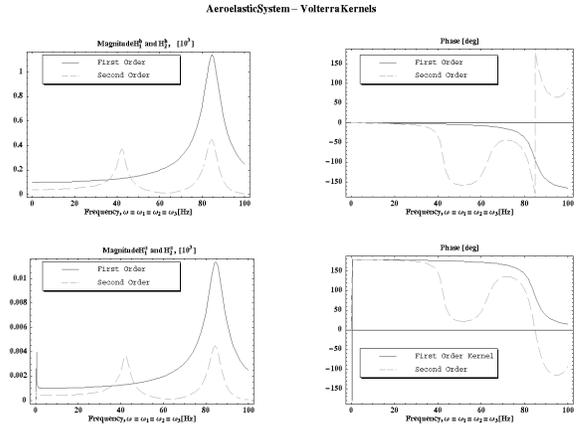
**Fig. 2** Convergence study involving the first three kernels and comparison with the “exact” nonlinear aeroelastic response to (a) *I-COSINE* gust pulse and (b) to *triangular* blast load, as shown in the inset

The excellent agreement of the predictions, assess both the accuracy of the aeroelastic model and also the power of the methodology based on the Volterra series and indicial function approach. The first, the second and the third approximations of the aeroelastic response to the two loads *1-COSINE* gust load and triangular blast load are plotted for different parameters, together with the “exact” response of the aeroelastic system as obtained through digital-computer solution of the nonlinear aeroelastic governing equations. Both figures reveal the rapid convergence of the approximation. The parameters in use for the simulations, unless otherwise specified, are chosen as: ( $m = 1; a = -0.2; c_{h1} = 10; (k_{h1} = 10^4; c_{\alpha 1} = 10; k_{\alpha 1} = 10^4; c_{h2} = 10; k_{h2} = 10^7; c_{\alpha 2} = 10; k_{\alpha 2} = 10^7; b = 1; \rho = 0.125; U_\infty = 0.4U_F; C_{L\alpha} = 2\pi)$ ). For the 2-D lifting surface encompassing pure plunging, the first three aeroelastic kernels in magnitude and phase are depicted in Fig. 3 as a function of the frequency, considering that  $\omega = \omega_1 = \omega_2 = \omega_3$ , i.e. the representation is given along the diagonal of the plane  $\omega_1, \omega_2$ . As is clearly seen, a reduced influence on the response of the third kernel is experienced.

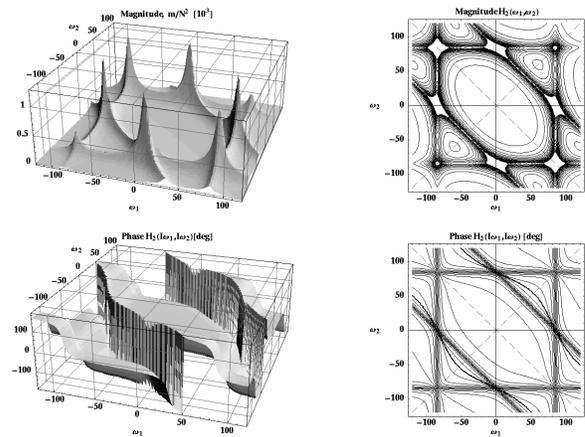


**Fig. 3** First three aeroelastic kernels of 2-D lifting surface, pure plunging motion

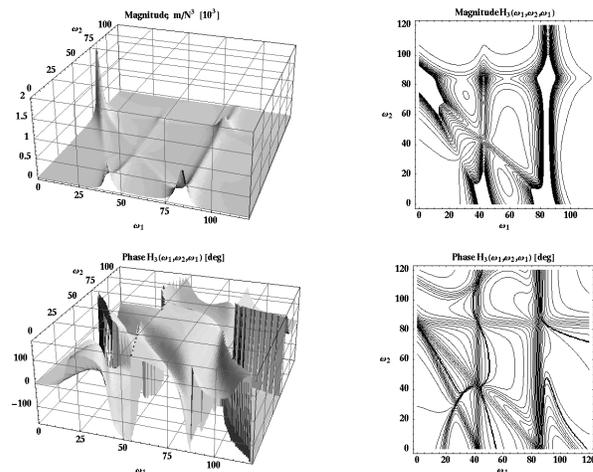
In Figs. 4 the Volterra’s kernels for the lifting surface featuring plunging - pitching coupled motions are depicted. Also in this case in the plots include the magnitude and phase for the kernels in plunging  $H_i^h$  and pitching  $H_i^\alpha$ , in which  $i$  identifies the order of the kernel.



**Fig. 4** First two aeroelastic kernels of 2-D lifting surface, plunging - pitching coupled motions



**Fig. 5** 3-D and contour plots of second order aeroelastic kernel



**Fig. 6** 3-D and contour plots of third order aeroelastic kernel

A 3-D plots of the magnitude and phase of the second order kernel vs. the two frequencies  $\omega_1$  and  $\omega_2$  are displayed in Fig. 5. The contour plots reveals the symmetry of this kernel respect to the diagonal represented by  $\omega_1 = \omega_2$ . In a 3-D plot, the third order Volterra kernel for the case in which  $\omega_3 = \omega_1$  are depicted in Figs. 6.

### Response in Time and Frequency Domains

Determination of subcritical aeroelastic response to any time-dependent externally applied load is useful in the design of wing structures and of the associated feedback control systems. In certain types of nonlinear analysis we are only interested in the special case considering of  $\tau_1 = \tau_2 = \dots = \tau_n = \tau$ <sup>34</sup>. This case can be represented as:

$$g(\tau) \equiv h_n(\tau_1, \tau_2, \dots, \tau_n)_{\tau_1 = \tau_2 = \dots = \tau_n = \tau} \quad (26)$$

This has a corresponding Laplace transform  $G(s)$  (so called *associated transform*) in the single-dimensional Laplace transform space:  $G(s) = \mathcal{L}[g(\tau)]$ . The response in time can be obtained from  $H(s_1, s_2, \dots, s_n)$  to find  $G(s)$  first and evaluate the single dimensional inverse Laplace transform  $g(\tau)$ .

This approach is called *association of variable*<sup>34</sup>. The nonlinear aeroelastic response in the time domain is depicted in Figs. 7 for a 2-D lifting surface featuring the plunging degree of freedom. In this figure the first plot represents the linear impulse response that corresponds to the convolution integral for the linear analysis. The other three plots represent the components of the response due to the second and the third order kernels and the total response as a combination of the three partial responses. The aeroelastic response will be presented and validated. The influence of the linear and nonlinear stiffness and the damping coefficients on the response, not displayed in this paper, reveals that, an increase in the damping coefficient contributes to the decrease of the response amplitude.

An increase of the nonlinear damping or of the stiffness coefficients contributes to the decrease of the magnitude of the kernels and consequently, of the response amplitude. This shows that the nonlinearities in the stiffness and damping play a beneficial role on the subcritical aeroelastic response<sup>15</sup>.

Figure 8 highlights the effect of the speed parameter  $V$  ( $\equiv U/b \omega_\alpha$ ) on the lifting surfaces subjected to sonic-boom pressure signature as shown in the inset.

### Nonlinear Aeroelastic Responses – Time Domain

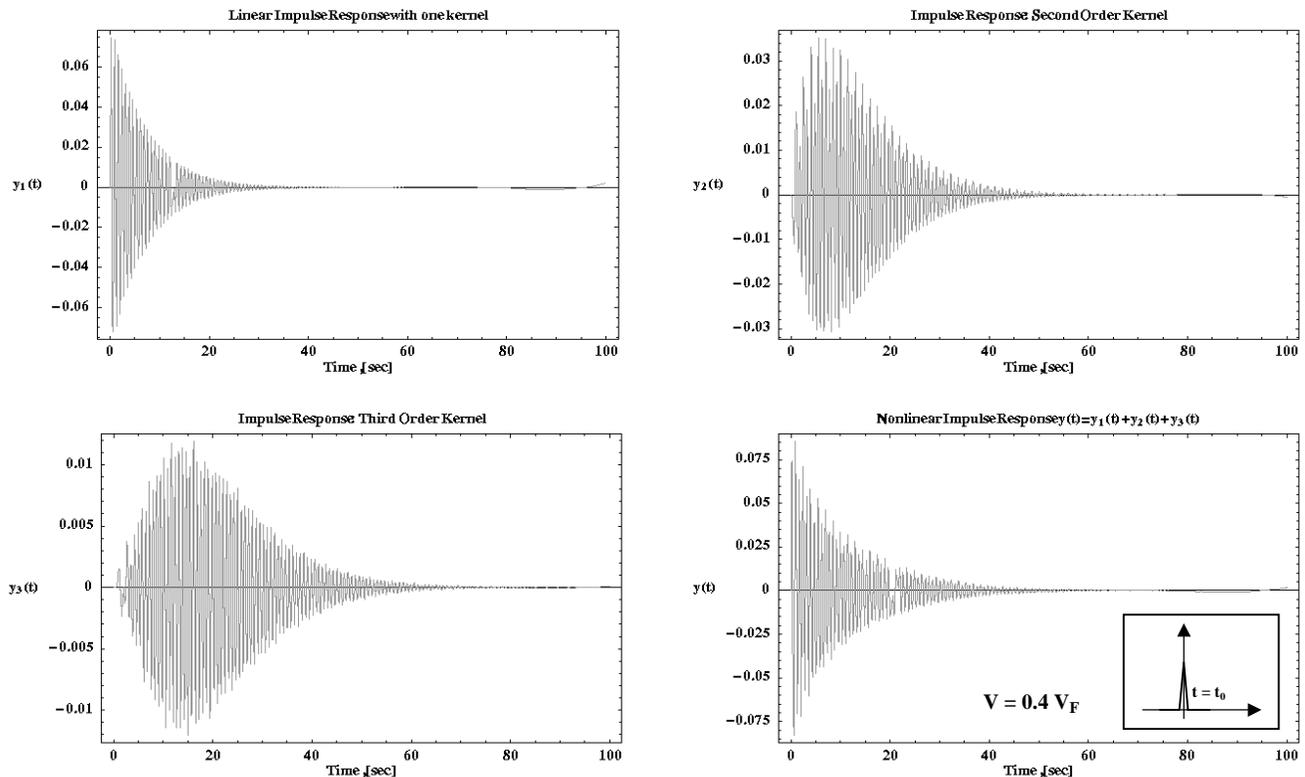
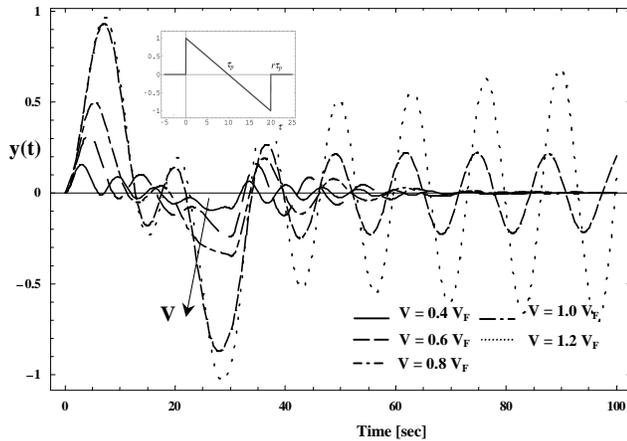


Fig. 7 Time-history of the nonlinear aeroelastic response



**Fig. 8** Influence of the flight speed on the nonlinear aeroelastic response to a *sonic-boom*, as shown in the inset ( $\tau_p = 15$  sec;  $r = 2$ ), evaluated with three kernels.

Herein  $\tau_p$  denotes the positive phase duration of the pulse measured from the time of impact of the structure;  $r$  denotes the shock pulse length factor. For  $r = 2$  the N-shaped pulse degenerates into a symmetric sonic-boom pulse, in the sense that its positive phase has the same characteristics as its negative one, and for  $r = 1$  a triangular pulse that corresponds to an explosive pulse is obtained. It becomes apparent that the amplitude of the response time-history (that have been evaluated for practical use with three kernels) increases with the increase of  $V$ . Moreover, in a certain range of speeds, as time unfolds, a decay of the amplitude is experienced, which reflects the fact that in this case the subcritical response is involved. However, for the dimensionless speed parameter  $V$  greater than the flutter speed (this one was determined using the linearized aeroelastic system), the response becomes unbounded implying that the occurrence of the flutter instability is impending. Also in this case the nonlinear stiffness and damping coefficients play a beneficial role on the subcritical aeroelastic response.

### CONCLUSIONS

Several issues that concern the nonlinear aeroelastic response via Volterra's series approach have been presented. It was also shown that, the method based on Volterra series opens large opportunities to approach in an unified and efficient way problems of nonlinear aeroelastic response and flutter. In addition, following the same approach, the character of the instability boundary, i.e. benign or catastrophic will also be addressed. This analysis will be done by using the concept of the first Liapunov quantity as developed by Bautin<sup>35</sup>.

Moreover, this approach can be extended as to include also active control capabilities. In spite of this, few of these potentialities have been explored yet. Comparisons of results carried out via Volterra series in conjunction with indicial functions approach and classical approach have been provided in Fig. 1 for the linearized model. It should also be stressed that aerodynamic indicial functions (for incompressible/compressible flow fields<sup>31</sup>) considered in conjunction with Volterra's series approach can be used as a powerful analytical tool for developing unsteady aerodynamic models and a unified nonlinear aeroelastic model. To the best of the authors' knowledge, with the exception of this paper, the problem of the aeroelastic response of lifting surfaces to external pulses via Volterra's series and indicial function approach was not yet addressed in the literature.

### ACKNOWLEDGMENT

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