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TERMINATION OF STRING REWRITING RULES THAT HAVE ONE PAIR OF OVERLAPS*

ALFONS GESER†

Abstract. This paper presents a partial solution to the long standing open problem of termination of one-rule string rewriting. Overlaps between the two sides of the rule play a central role in existing termination criteria. We characterize termination of all one-rule string rewriting systems that have one such overlap at either end. This both completes a result of Kurth and generalizes a result of Shikishima-Tsuji et al.

Key words. semi-Thue system, string rewriting, one-rule, single-rule, termination, uniform termination, overlap

Subject classification. Computer Science

1. Introduction and Related Work. Termination of one-rule string rewriting systems (SRSs) is a long standing open problem [12, 13, 11, 15, 14, 7, 16, 18, 2, 3, 4]. The first systematic approach was started by Kurth [8]. He introduced a number of termination criteria to solve termination for all $\ell \to r$ where $|r| \le 6$.[1]

Most of Kurth's criteria (5 out of 8), and indeed most of the criteria introduced since, are based on two sets: the set of overlaps of the left hand side (from the left end) with the right hand side (from the right end); and the set of overlaps of the right hand side (from the left end) with the left hand side (from the right end). Kurth's Criterion D states that we have termination if one or both of the two sets are empty.

In the case where both sets are singletons, we say that the one-rule SRS has *one pair of overlaps*. Kurth [8] provides Criterion F specifically for this case. As Criterion F can only prove termination of rules that are left barren or right barren, it is incomplete as we will show (Example 2). Shikishima-Tsuji et al. [16, Theorem 2] show that a *confluent* one-rule SRS with one pair of overlaps terminates if and only if there are no loops of lengths 1 or 2. As a consequence termination of such SRSs is decidable.

This paper completely solves the termination problem for one-rule SRSs with one overlap pair. We prove that such an SRS terminates if and only if it has no loop of lengths 1, 2 or 3 (Theorem 7.1). This implies decidability of the termination problem.

It turns out that the extension is non-trivial. There are two behaviours that were observed neither by Kurth nor by Shikishima-Tsuji et al. Loops of length 3 is one of them; the other is terminating non-tame rules.

This paper makes the following original contributions:

1. Termination of one-rule SRSs with one overlap pair is shown decidable.
2. Termination of one-rule SRSs with one overlap pair is shown equivalent to the non-existence of loops of length 3 or less.
3. Terminating one-rule SRSs with one overlap pair are shown to have linear derivation lengths.
4. The first termination criterion for a class of non-tame one-rule SRSs.

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[1] An English presentation of Kurth's chapter on termination can be found in the author's habilitation thesis [3].

The paper is organized as follows. After the preliminaries (Section 2) and an introduction to left barren and tame rules (Section 3), we focus on the interesting non-tame case. In Section 4, we derive a pattern that describes the non-tame rules. In Sections 5 and 6, we solve the non-terminating and terminating non-tame rules, respectively. Section 7 finally shows the main theorem of the paper and its ramifications.

2. Preliminaries. A *string rewriting rule* is a pair $\ell \to r$ of strings, $\ell, r \in \Sigma^*$ where Σ is a given alphabet. A set of string rewriting rules is called a *string rewriting system* (SRS). An SRS R induces a *rewrite step* relation $\to$ defined by $s \to t$ if there are $u, v \in \Sigma^*$ and a rule $\ell \to r$ in R such that $s = u\ell v$ and $t = urv$. The SRS R is said to *terminate* if there is no infinite sequence of rewrite steps $s_1 \to s_2 \to \dots$.

A string u is called a *factor* of v if $v = sut$ for some $s, t \in \Sigma^*$; a *prefix* if $v = ut$ for some $t \in \Sigma^*$; a *suffix* if $v = su$ for some $s \in \Sigma^*$. The prefix or suffix u of v is called *proper* if $u \neq v$. The set of *overlaps* of a string u with a string v is defined by

$$\mathrm{OVL}(u, v) = \{w \in \Sigma^+ \mid u = u'w, v = wv', u'v' \neq \varepsilon, u', v' \in \Sigma^*\}.$$

The length of a string u is denoted by $|u|$.

3. Left Barren Rules. For a fixed one-rule SRS $\{\ell \to r\}$ let $A = \mathrm{OVL}(r, \ell)$ and $B = \mathrm{OVL}(\ell, r)$. In what follows we consider A and B as disjoint. For all $\alpha \in A$, the strings ℓ_α and r_α are defined by $\ell = \alpha\ell_\alpha$ and $r = r_\alpha\alpha$, respectively. Likewise, for all $\beta \in B$, the strings ℓ_β and r_β are defined by $\ell = \ell_\beta\beta$ and $r = \beta r_\beta$, respectively.

The following definition of "left barren" is after McNaughton's corrected version. The original definition is renamed to "left s-barren" (see Definition 3.4), following a suggestion of Kobayashi et al. [7].

DEFINITION 3.1 (Left barren, right barren [12]). *A one-rule SRS $\{\ell \to r\}$ is called* left barren *if ℓ is not a factor of r and no $\ell_\alpha, \alpha \in A$ is a prefix of any concatenation $r_{\beta_1} \dots r_{\beta_k}$ where $\beta_1, \dots, \beta_k \in B, k \geq 1$. Dually, $\{\ell \to r\}$ is called* right barren *if ℓ is not a factor of r and no $\ell_\beta, \beta \in B$ is a suffix of any concatenation $r_{\alpha_1} \dots r_{\alpha_k}$ where $\alpha_1, \dots, \alpha_k \in A, k \geq 1$.*

A one-rule SRS $\{\ell \to r\}$ is called *non-overlapping* if $\mathrm{OVL}(\ell, \ell) = \emptyset$.

THEOREM 3.2 ([12]). *Every non-overlapping, left barren, one-rule SRS terminates.*

THEOREM 3.3 ([3]). *Every left barren one-rule SRS terminates.*

By symmetry w.r.t. reversal of strings also every right barren one-rule SRS terminates.

DEFINITION 3.4 (Left s-barren, right s-barren [12, 7]). *A rule $\ell \to r$ is called* left s-barren *if no $\ell_\alpha, \alpha \in A$ is a prefix of any $r_\beta, \beta \in B$. Dually $\ell \to r$ is called* right s-barren *if no $\ell_\beta, \beta \in B$ is a suffix of any $r_\alpha, \alpha \in A$.*

A left barren rule is left s-barren, but the converse usually does not hold. Indeed we will encounter left s-barren, not left barren rules later in this paper. They belong to a class of rules whose termination is particularly difficult to show. Next we will define this class.

In the following definition we consider A, B as (disjoint) alphabets. For $\overline{\alpha} = \alpha_1\alpha_2 \dots \alpha_k \in A^*$ we define $\ell_{\overline{\alpha}}$ by $\ell_{\overline{\alpha}} = \ell_{\alpha_1}\ell_{\alpha_2} \dots \ell_{\alpha_k}$. And dually, for $\overline{\beta} = \beta_1\beta_2 \dots \beta_k \in B^*$ we define $\ell_{\overline{\beta}}$ by $\ell_{\overline{\beta}} = \ell_{\beta_1}\ell_{\beta_2} \dots \ell_{\beta_k}$.

Kobayashi et al. [7] introduced the notion of tame, non-overlapping one-rule SRSs.

DEFINITION 3.5 (Tame [3]). *Let $\{\ell \to r\}$ be a one-rule SRS. The sets C and D are defined by*

$$C = \{r' \in \Sigma^* \mid r = \beta\ell_{\overline{\alpha}}r', \beta \in B, \overline{\alpha} \in A^*\},$$
$$D = \{r' \in \Sigma^* \mid r = r'\ell_{\overline{\beta}}\alpha, \alpha \in A, \overline{\beta} \in B^*\}.$$

Then $\ell \to r$ is called tame *if ℓ is neither of the form*

$$\alpha r_1 r_2 \dots r_k w, \tag{3.1}$$

for any $\alpha \in A$, $k \geq 1$, $r_1, \ldots, r_k \in C$, and non-empty prefix w of an element of C; nor of the form

$$wr_1 r_2 \ldots r_j \beta, \tag{3.2}$$

for any $\beta \in B$, $j \geq 1$, $r_1, \ldots, r_j \in D$, and non-empty suffix w of an element of D.

The following result is implicit in Kobayashi et al. [7, Cor. 5.9].

THEOREM 3.6. *Every non-overlapping, tame, left s-barren one-rule SRS is left barren.*

THEOREM 3.7 ([3]). *Every tame, left s-barren one-rule SRS is left barren.*

By symmetry, every tame, right s-barren one-rule SRS is right barren.

Proof. For a proof by contradiction, assume that $\ell \to r$ is not left barren, i.e., some ℓ_α is a prefix of some concatenation $r_{\beta_1} r_{\beta_2} \cdots r_{\beta_n}$. Let n be minimal. If $n = 1$ then $\ell \to r$ is not left s-barren. So $n \geq 2$ whence ℓ_α is of the form $r_{\beta_1} r_{\beta_2} \cdots r_{\beta_{n-1}} w$ where w is a nonempty prefix of r_{β_n}. Hence ℓ is of the form (3.1) and so $\ell \to r$ is not tame. □

4. A Reduction of the Problem. Throughout the remainder of this paper we assume a one-rule SRS $\{\ell \to r\}$ that has one pair of overlaps, i.e., $|\operatorname{OVL}(r, \ell)| = |\operatorname{OVL}(\ell, r)| = 1$. Let then $\alpha, \beta \in \Sigma^+$ be defined by $\operatorname{OVL}(r, \ell) = \{\alpha\}$ and $\operatorname{OVL}(\ell, r) = \{\beta\}$.

We will devote the greater part of the paper to solving the interesting case: rules that are left s-barren but neither left barren nor right s-barren. According to Theorem 3.7, these are non-tame, specifically they are of the form (3.1). In this section we will derive the general pattern of such rules. Let us henceforth assume that ℓ is not a factor of r and that $|\ell| < |r|$.

The first pattern is derived without the right-s-barren hypothesis.

LEMMA 4.1. *Let $\ell \to r$ be left s-barren but not left barren. Then $|\beta| > |\alpha|$ and $\ell \to r$ is of the form*

$$\alpha(ww')^{n-1}w \to \beta ww' \tag{4.1}$$

for some $n \geq 2$, $w' \in \Sigma^*$, and $w \in \Sigma^+$.

Proof. Let $\ell \to r$ be left s-barren but not left barren. Then we get by the respective definitions that ℓ_α is not a prefix of r_β and that ℓ_α is a prefix of r_β^n form some $n \geq 1$. Hence r_β is a proper prefix of ℓ_α. So let $\ell_\alpha = r_\beta^{n-1} w$ where $n \geq 2$, and w is a non-empty prefix of r_β. Let $w' \in \Sigma^*$ be defined by $r_\beta = ww'$. By back-substitution we get the form (4.1). From $|\beta r_\beta| = |r| > |\ell| = |\alpha r_\beta^{n-1} w|$ we conclude $|\beta| > |\alpha|$. □

If we add the right-s-barren hypothesis, then we can rule out the case where α and β overlap in ℓ.

LEMMA 4.2. *If $\ell \to r$ is left s-barren but neither left barren nor right s-barren, then $|\alpha| + |\beta| \leq |\ell|$.*

Proof. For a proof by contradiction assume $|\alpha| + |\beta| > |\ell|$. Let $\ell \to r$ be left s-barren but not left barren. By Lemma 4.1 we get that $\ell \to r$ has the form (4.1). Then by $|\alpha| + |\beta| > |\ell|$ there is a non-empty suffix u of α such that $\beta = u(ww')^{n-1}w$. Define $\alpha' \in \Sigma^*$ by $\alpha = \alpha' u$. The string α' is non-empty by $\beta \neq \ell$. Thus ℓ and r are of the form

$$\begin{aligned} \ell &= \alpha' u (ww')^{n-1} w, \\ r &= u(ww')^{n-1} www', \end{aligned}$$

for some $n \geq 2$, $w' \in \Sigma^*$, and $\alpha', u, w \in \Sigma^+$.

Now let moreover $\ell \to r$ not be right s-barren, i.e., let ℓ_β be a suffix of r_α. This is expressed equivalently by the string equation $z\ell_\beta \alpha = r$ for some $z \in \Sigma^*$. Using $\ell_\beta = \alpha'$ this instantiates to

$$z\alpha'\alpha' u = u(ww')^{n-1} www'.$$

Let $m \geq 0$ be maximal such that $((ww')^{n-1}www')^m$ is a suffix of u. Define $u_1 \in \Sigma^*$ by $u = u_1((ww')^{n-1}www')^m$. Then u_1 is a proper suffix of $(ww')^{n-1}www'$, and the equation reduces to $z\alpha'\alpha'u_1 = u_1(ww')^{n-1}www'$. If $m > 0$ then $\alpha'u_1 \in \mathrm{OVL}(r, \ell)$, a contradiction. So $m = 0$ and $u = u_1$.

If u_1 is a suffix of ww' then $u_1w \in \mathrm{OVL}(\ell, r)$, a contradiction. So ww' is a proper suffix of u_1. Let $u_2 \in \Sigma^+$ be defined by $u_1 = u_2ww'$. The equation reduces to $z\alpha'\alpha'u_2 = u_2(ww')^nw$.

By definition of u_1, u_2 is a proper suffix of $(ww')^{n-1}w$. Then $u_2 \in \mathrm{OVL}(\ell, r)$, a contradiction. $\square$

If α and β do not overlap in ℓ, then we can narrow the pattern for the rule:

Lemma 4.3. *Let $\ell \to r$ be left s-barren but not left barren. If $|\alpha| + |\beta| \leq |\ell|$ then $\ell \to r$ is of the form*

$$\alpha wxy\alpha w \to y\alpha wwxy\alpha \tag{4.2}$$

for some $x \in \Sigma^*$ and $y, \alpha, w \in \Sigma^+$.

Proof. Let $\ell \to r$ be left s-barren but not left barren. By Lemma 4.1 we get that $\ell \to r$ has the form (4.1).

Case 1: $\beta = w''(w'w)^i$ for some $0 \leq i \leq n-1$, and some non-empty suffix w'' of w. If $i \geq 1$ then $w'' \in \mathrm{OVL}(\ell, r)$, a contradiction. So $i = 0$ and $\beta = w''$. Then

$$|r| - |\ell| = |w''| + |w| + |w'| - (|\alpha| + n|w| + (n-1)|w'|) < 0,$$

again a contradiction.

Case 2: $\beta = w''w(w'w)^i$ for some $0 \leq i \leq n-2$, and some nonempty suffix w'' of w'. If $i \geq 1$ then $w''w \in \mathrm{OVL}(\ell, r)$, a contradiction. So $i = 0$ and $\beta = w''w$. Let $w' = xw''$ for some string x. Then we have

$$\begin{aligned} \ell &= \alpha(wxw'')^{n-1}w, \\ r &= w''wwxw'', \end{aligned}$$

and so

$$\begin{aligned} |r| - |\ell| &= 2|w''| + 2|w| + |x| - (|\alpha| + (n-1)|w''| + (n-1)|x| + n|w|) \\ &= (3-n)|w''| + (2-n)|w| + (2-n)|x| - |\alpha|. \end{aligned}$$

If $n \geq 3$ then $|r| - |\ell| < 0$. So $n = 2$ and $|r| - |\ell| = |w''| - |\alpha| > 0$ whence $|w''| > |\alpha|$. By definition of α now α is a proper suffix of w''. Let $w'' = y\alpha$ for some $y \in \Sigma^+$. We conclude that $\ell \to r$ is of the form (4.2). $\square$

Putting Lemma 4.2 and 4.3 together allows us to narrow the rule pattern further:

Lemma 4.4. *If $\ell \to r$ is left s-barren but neither left barren nor right s-barren then $\ell \to r$ is of the form*

$$\alpha wx(y\alpha wx)^{m+1}\alpha w \to y\alpha wx\alpha wwx(y\alpha wx)^{m+1}\alpha. \tag{4.3}$$

for some $m \geq 0$, $x \in \Sigma^*$, and $\alpha, w, y \in \Sigma^+$.

Proof. Let $\ell \to r$ be left s-barren but neither left barren nor right s-barren. By Lemma 4.2 we get $|\alpha| + |\beta| \leq |\ell|$. By Lemma 4.3 we get that $\ell \to r$ has the form (4.2).

The property that $\ell \to r$ is not right s-barren means that $\ell_\beta = \alpha wx$ is a suffix of $r_\alpha = y\alpha wwxy$. Then we have to solve the string equation

$$z\alpha wx = y\alpha wwxy \tag{4.4}$$

for $z, x \in \Sigma^*, \alpha, w, y \in \Sigma^+$.

Let $m \geq 0$ be maximal such that y^m is a suffix of x. Define $x_1 \in \Sigma^*$ by $x = x_1y^m$. Then $z\alpha wx_1 = y\alpha wwx_1y$ and x_1 is a proper suffix of y. Define $y_1 \in \Sigma^+$ by $y = y_1x_1$. Then $z\alpha w = y_1x_1\alpha wwx_1y_1$.

If y_1 is a suffix of w then $y_1 \in \mathrm{OVL}(\ell, r)$, a contradiction. So w is a proper suffix of y_1. Define $y_2 \in \Sigma^+$ by $y_1 = y_2 w$. Then the equation reduces to $z\alpha = y_2 w x_1 \alpha w w x_1 y_2$.

If y_2 is a suffix of α then $y_2 w \in \mathrm{OVL}(\ell, r)$, a contradiction. So α is a proper suffix of y_2. Define $y_3 \in \Sigma^+$ by $y_2 = y_3 \alpha$. The equation reduces to $z = y_3 \alpha w x_1 \alpha w w x_1 y_3$ which is trivial.

By back-substitution we get

$$\begin{aligned}
y &= y_1 x_1 = y_2 w x_1 = y_3 \alpha w x_1,\\
x &= x_1 y^m = x_1 (y_3 \alpha w x_1)^m,\\
\ell &= \alpha w x y \alpha w = \alpha w x_1 (y_3 \alpha w x_1)^{m+1} \alpha w,\\
r &= y \alpha w w x y \alpha = y_3 \alpha w x_1 \alpha w w x_1 (y_3 \alpha w x_1)^{m+1} \alpha.
\end{aligned}$$

and thus the form (4.3) by the renaming $x_1 \mapsto x, y_3 \mapsto y$. □

The following is interesting to note. It explains why rules of the form (4.3) were not observed by Shikishima-Tsuji et al.

THEOREM 4.5. *All rules of the form* (4.3) *are non-confluent.*

Proof. A one-rule SRS $\{\ell \to r\}$ where $|\ell| < |r|$ is confluent if and only if $\mathrm{OVL}(\ell, \ell) \subseteq \mathrm{OVL}(r, r)$ by a result of Wrathall [17]. A rule of the form (4.3) satisfies $\alpha w \in \mathrm{OVL}(\ell, \ell)$. If $\alpha w \in \mathrm{OVL}(r, r)$ then $\alpha w \in \mathrm{OVL}(r, \ell)$, a contradiction to $\mathrm{OVL}(r, \ell) = \{\alpha\}$. So $\alpha w \in \mathrm{OVL}(\ell, \ell) \setminus \mathrm{OVL}(r, r)$ whence $\ell \to r$ is not confluent. □

In the next two sections we are going to identify the non-terminating and the terminating instances of the form (4.3).

5. The Non-terminating Case. A rule of the form (4.3) loops in the following case:

LEMMA 5.1. *Let $\ell \to r$ be left s-barren but neither left barren nor right s-barren. If $\ell_\beta \ell_\beta$ is a suffix of r_α, then the one-rule SRS $\{\ell \to r\}$ has a loop of length 3.*

Proof. Like in the proof of Lemma 4.1, we get $\ell_\alpha = r_\beta^{n-1} w$ and $r_\beta = w w'$ for some $w \in \Sigma^+, w' \in \Sigma^*, n \geq 2$. In the proof of Lemma 4.3 we showed $n = 2$. With $r_\alpha = v \ell_\beta \ell_\beta$ for some $v \in \Sigma^*$, we then get a loop:

$$\begin{aligned}
\ell \ell_\alpha \to r_\alpha \alpha \ell_\alpha \to r_\alpha r = v \ell_\beta \ell_\beta \beta r_\beta \to v \ell_\beta r r_\beta = v \ell_\beta \beta r_\beta r_\beta = v \ell r_\beta r_\beta\\
= v \ell \ell_\alpha w'.
\end{aligned}$$

□

These loops are also instances of Kurth's criterion for loops of length 3 [9, Theorem 2, Case A]. The following little result provides an alternative criterion to Lemma 5.1.

LEMMA 5.2. *If $\ell \to r$ has the form* (4.3) *then the following are equivalent:*

1. *$\ell_\beta \ell_\beta$ is a suffix of r_α,*
2. *$m = 0$ and $y = y' \alpha w x$ for some $y' \in \Sigma^+$.*

Proof. Obviously (2) implies (1). Next we show the converse by contradiction. Let $\ell \to r$ have the form (4.3) and let $\ell_\beta \ell_\beta$ be a suffix of r_α. Define $v \in \Sigma^*$ by $r_\alpha = v \ell_\beta \ell_\beta$. If $m > 0$ then y is a suffix of $y \alpha w$ and then $y \alpha w \in \mathrm{OVL}(\ell, r)$, a contradiction. With $m > 0$, the string $\alpha w x$ is a suffix of $\alpha w w x y$. If y is a suffix of $\alpha w x$ then $y \alpha w \in \mathrm{OVL}(\ell, r)$, a contradiction. So $\alpha w x$ is a proper suffix of y, i.e., there is $y' \in \Sigma^+$ such that $y = y' \alpha w x$. □

EXAMPLE 1. *The one-rule SRS*

$$abdababab \to dabababbdababa$$

has a loop of length 3:

$$\begin{aligned}&\underline{abdababab}bdababab \rightarrow\\&dababab bdabab\underline{abdababab} \rightarrow\\&dabababbdab\underline{abdababab}bdababa \rightarrow\\&dabababbd\boxed{abdabababbdababab}dababa.\end{aligned}$$

Redexes are underlined. The re-occurrence of the start string is indicated by a box. This example provides the smallest non-terminating witness ($|r| = 14$) *of Lemma 4.4.*

6. The Terminating Case. For this section let us assume a rule of the form (4.3) where $\ell_\beta \ell_\beta$ is not a suffix of r_α. We are going to reduce termination of such a rule to termination of an SRS R over a different alphabet. Termination of R will be easy to prove.

Define r_δ, $r_{\beta,\alpha}$, and $r_{\beta,\delta}$ by

$$r = r_\delta \ell_\beta \alpha, \qquad r = \beta r_{\beta,\alpha} \alpha, \qquad r = \beta r_{\beta,\delta} \ell_\beta \alpha.$$

These definitions are sound as witnessed by

$$\begin{aligned}\beta &= y\alpha wx\alpha w,\\ \ell_\beta &= \alpha wx(y\alpha wx)^m,\\ r_\delta &= y\alpha wx\alpha wwxy,\\ r_{\beta,\alpha} &= wx(y\alpha wx)^{m+1},\\ r_{\beta,\delta} &= wxy.\end{aligned}$$

Lemma 6.1. *Let $\ell \rightarrow r$ have the form* (4.3)*. Then the following rewrite steps exist:*

$$\begin{array}{lll} r_\alpha r \rightarrow_{\ell\rightarrow r} r_\delta r r_\beta, & r_\alpha r_\alpha \rightarrow_{\ell\rightarrow r} r_\delta r r_{\beta,\alpha}, & r_\alpha r_\delta \rightarrow_{\ell\rightarrow r} r_\delta r r_{\beta,\delta},\\ r_{\beta,\alpha} r \rightarrow_{\ell\rightarrow r} r_{\beta,\delta} r r_\beta, & r_{\beta,\alpha} r_\alpha \rightarrow_{\ell\rightarrow r} r_{\beta,\delta} r r_{\beta,\alpha}, & r_{\beta,\alpha} r_\delta \rightarrow_{\ell\rightarrow r} r_{\beta,\delta} r r_{\beta,\delta}.\end{array}$$

Proof. *Routine.* □

Lemma 6.2. *Let $\ell \rightarrow r$ have the form* (4.3) *and let $\ell_\beta \ell_\beta$ not be a suffix of r_α. Then ℓ is not a factor of any of the following: (1) $r_\delta^i r$, (2) $r r_\beta$, (3) $r r_{\beta,\delta} r_\delta^i r$ for any $i \geq 0$.*

Proof. For Claim 1, let $i \geq 1$ be least such that ℓ is a factor of $r_\delta^i r$. Then ℓ_β is a suffix of r_δ^i because β is the only overlap of ℓ with r. Since $\ell_\beta \ell_\beta$ is not a suffix of $r_\alpha = r_\delta \ell_\beta$, ℓ_β is not a suffix of y. Hence y is a proper suffix of ℓ_β and so of $y\alpha wx$. So $y\alpha w \in \mathrm{OVL}(\ell, r)$, a contradiction.

For Claim 2, let ℓ be a factor of $r r_\beta$. Because α is the only overlap between r and ℓ, we have $|\ell_\alpha| \leq |r_\beta|$, a contradiction.

For Claim 3 assume that ℓ is a factor of $r r_{\beta,\delta} r_\delta^i r$ for some $i \geq 0$. By Claims 1 and 2, ℓ is neither a factor of $r_{\beta,\delta} r_\delta^i r$ nor of $r r_{\beta,\delta}$; so ℓ is of the form $\ell' r_{\beta,\delta} r_\delta^j \ell''$ for some $0 \leq j \leq i$ and some non-empty suffix ℓ' of r and some non-empty prefix ℓ'' of r. Thus ℓ is of the form $\alpha r_{\beta,\delta} r_\delta^j \beta$. If $j = 0$ then $wx(y\alpha wx)^m = wxy$ which contradicts $y, \alpha \in \Sigma^+$. So $j > 0$ and y is a proper suffix of ℓ_β. We get a contradiction by $y\alpha w \in \mathrm{OVL}(\ell, r)$. □

The six-rule SRS R over $\Omega = \{a, b, c, d, e, f\}$ is defined as follows:

$$\begin{aligned}R = \{g'g'' \rightarrow h'fh'' \mid\ & (g', h') \in \{(a, d), (c, e)\},\\ & (g'', h'') \in \{(a, c), (d, e), (f, b)\}\}\end{aligned}$$

Define the weight $wt^*(x)$ of a string x by $wt(a) = wt(c) = 3$, $wt(b) = wt(d) = wt(e) = wt(f) = 1$, and $wt^*(x_1 \dots x_k) = \sum_{i=1}^{k} wt(x_i)$. Then R terminates by

$$wt^*(u) - wt^*(v) = (wt(g') - wt(h')) - wt(f) + (wt(g'') - wt(h'')) = 2 - 1 + 0 > 0$$

for all rewrite steps $u \to_R v$.

Let the string homomorphism $\phi : \Omega^* \to \Sigma^*$ be defined by $\phi(a) = r_\alpha, \phi(b) = r_\beta, \phi(c) = r_{\beta,\alpha}, \phi(d) = r_\delta, \phi(e) = r_{\beta,\delta}, \phi(f) = r$. By Lemma 6.1, $u \to_R v$ implies $\phi(u) \to_{\ell\to r} \phi(v)$ for all $u, v \in \Omega^*$. However we will need the converse direction. To this end let us define the regular language $\mathcal{M}$ by

$$\mathcal{M} = (a + d(fe)^* + d(fe)^* fc)^* (af + d(fe)^* f(cf + b)) + f.$$

Let $\phi[\mathcal{M}]$ denote the set $\{\phi(u) \mid u \in \mathcal{M}\}$. We are going to show that $\{\ell \to r\}$-reduction steps on $\phi[\mathcal{M}]$ can be simulated by R-reduction steps. First we show that R-reduction preserves $\phi[\mathcal{M}]$.

LEMMA 6.3. *If $u \in \mathcal{M}$ and $u \to_R v$ then $v \in \mathcal{M}$.*

Proof. Let $(g', h') \in \{(a, d), (c, e)\}$ and $(g'', h'') \in \{(a, c), (d, e), (f, b)\}$. Let $u = u'g'g''u'' \in \mathcal{M}$ and $v = u'h'fh''u''$. Then we derive

$$\begin{aligned} &u' \in (a + d(fe)^* + d(fe)^* fc)^* && \text{if } g' = a, \\ &u' \in (a + d(fe)^* + d(fe)^* fc)^* d(fe)^* f && \text{if } g' = c. \end{aligned}$$

Case 1: $g'' = a$. If $g' = a$ then $u'' \in \mathcal{M}$ whence $v = u'dfcu'' \in \mathcal{M}$. If $g' = c$ then $u'' = f$ whence $v = u'efcu'' \in \mathcal{M}$.

Case 2: $g'' = d$. Then

$$\begin{aligned} u'' \in\ &((fe)^* + (fe)^* fc)(a + d(fe)^* + d(fe)^* fc)^* (af + d(fe)^* f(cf + b)) \\ &+ (fe)^* f(cf + b). \end{aligned}$$

If $g' = a$ then $v = u'dfeu'' \in \mathcal{M}$. If $g' = c$ then $v = u'efeu'' \in \mathcal{M}$.

Case 3: $g'' = f$. If $g' = a$ then u'' is the empty string and $v = u'dfbu'' \in \mathcal{M}$. If $g' = c$ then u'' is again the empty string and $v = u'efbu'' \in \mathcal{M}$. ☐

Next we derive a few properties of $u \in \mathcal{M}$ if $\phi(u)$ contains a factor ℓ.

LEMMA 6.4. *Let $u \in \mathcal{M}$ and $s', s'' \in \Sigma^*$. If $\phi(u) = s'\ell s''$ then $u = u'g'g''u''$, $|\phi(u')| \le |s'| < |\phi(u'g')|$, $|\phi(u'')| \le |s''| < |\phi(g''u'')|$ for some $u', u'' \in \Omega^*$, $g' \in \{a, c\}$, $g'' \in \{a, d, f\}$.*

Proof. Suppose that $u \in \mathcal{M}$, $s', s'' \in \Sigma^*$, and $\phi(u) = s'\ell s''$. Let $u' \in \Omega^*$ be the longest prefix of u such that $|\phi(u')| \le |s'|$. Let $u'' \in \Omega^*$ be the longest suffix of u such that $|\phi(u'')| \le |s''|$. By $|\phi(u)| > |\phi(u'u'')|$ there is $v \in \Sigma^+$ such that $u = u'vu''$. Define $t', t'' \in \Sigma^*$ by $s' = \phi(u')t'$ and $s'' = t''\phi(u'')$. Then

$$\phi(u) = \phi(u')\phi(v)\phi(u'') = \phi(u')t'\ell t''\phi(u''),$$

whence $\phi(v) = t'\ell t''$. The case $|v| = 1$ implies that ℓ is a factor of r, so $|v| \ge 2$. We distinguish cases on the form of v.

Case 1: $v \in \Omega^*(a + c)(a + d + f)\Omega^*$. Let $g' \in \{a, c\}$, $g'' \in \{a, d, f\}$, $v', v'' \in \Omega^*$, and let $v = v'g'g''v''$. We further distinguish cases whether v', v'' are empty strings or not.

Case 1.1: $|v'| = |v''| = 0$. Then $v = g'g''$. By definition of u' we get $|t'| < |\phi(g')|$. By definition of u'' we get $|t''| < |\phi(g'')|$. The claim follows.

Case 1.2: $|v'| = 0$, $|v''| > 0$. By $|r| > |\ell|$ and $|r_\alpha| > |\ell|$ and $u \in \mathcal{M}$ we get $v \in (a+c)d^+(a+d+f)$. Let $v = v_0 g_0$ for some $v_0 \in (a+c)d^+$, and $g_0 \in \{a,d,f\}$. Then there are $\ell', \ell'' \in \Sigma^+$ such that $\ell = \ell'\ell''$, $\phi(v_0) = t'\ell'$, and $\phi(g_0) = \ell'' t''$. Since $\phi(g_0)$ is a prefix of r, we obtain $\ell'' \in \mathrm{OVL}(\ell, r)$, so $\ell'' = \beta$ and $\ell' = \ell_\beta$. By definition of v_0, now $\phi(d) = r_\delta = y\alpha wx\alpha wwxy$ is a suffix of $\ell_\beta = \alpha wx(y\alpha wx)^m$. So $m > 0$ and y is a suffix of $y\alpha wx$. Then $y\alpha w \in \mathrm{OVL}(\ell, r)$, a contradiction.

Case 1.3: $|v'| > 0$, $|v''| = 0$. Let $v = v_0 g_0$ for some $v_0 \in \Omega^+(a+c)$, and $g_0 \in \{a,d,f\}$. Then there are $\ell', \ell'' \in \Sigma^+$ such that $\ell = \ell'\ell''$, $\phi(v_0) = t'\ell'$, and $\phi(g_0) = \ell'' t''$. Since $\phi(g_0)$ is a prefix of r, we obtain $\ell'' \in \mathrm{OVL}(\ell, r)$, so $\ell'' = \beta$ and $\ell' = \ell_\beta$. Then

$$|\ell_\beta| = |\phi(v_0)| > |\phi(c)| = |r_{\beta,\alpha}| > |\ell_\beta|,$$

a contradiction.

Case 1.4: $|v'|, |v''| > 0$. By $|r| > |\ell|$ and $|r_\alpha| > |\ell|$ and $u \in \mathcal{M}$ we get $g' = c$ and $g'' = d$. So $\phi(cd) = r_{\beta,\alpha} r_\delta$ is a factor of ℓ, whence $|r_{\beta,\alpha} r_\delta| \le |\ell|$, a contradiction.

Case 2: $v \in \Omega^+ \setminus \Omega^*(a+c)(a+d+f)\Omega^*$. Define the set of fragments $\mathcal{F}(z)$ of a string $z \in \Omega^*$ as follows. If $z \in (\Omega \setminus \{f\})^*$ then $\mathcal{F}(z) = \{z\}$. Else $z = z_0 f z_1 \ldots f z_n$ for some $n \ge 1$ and unique $z_1, \ldots, z_n \in (\Omega \setminus \{f\})^*$; then

$$\mathcal{F}(z) = \{z_1 f, f z_2 f, \ldots, f z_{n-1} f, f z_n\}.$$

From $u \in \mathcal{M}$ then

$$\mathcal{F}(u) \in (a+d)^* f + f(e+c)(a+d)^* f + fb.$$

Because $|r| > |\ell|$, and ℓ is not a factor of r, we obtain $v \in \mathcal{F}(u)$. So

$$v \in \mathcal{F}(u) \setminus \Omega^*(a+c)(a+d+f)\Omega^* = d^* f + fed^* f + fb.$$

By Lemma 6.2, $\phi(v)$ has no factor ℓ, so this case is void. ☐

Now we are ready to state the simulation lemma.

Lemma 6.5. *Let $u \in \mathcal{M}$ and $t \in \Sigma^*$. If $\phi(u) \to_{\ell\to r} t$ then $\phi(v) = t$ and $u \to_R v$ for some $v \in \mathcal{M}$.*

Proof. Let $u \in \mathcal{M}$ and $s', s'', t \in \Sigma^*$, and let $\phi(u) = s'\ell s''$ and $t = s' r s''$. By Lemma 6.4 there are $u', u'' \in \Omega^*$, $g' \in \{a, c\}$, $g'' \in \{a, d, f\}$ such that $u = u' g' g'' u''$ and $|\phi(u')| \le |s'| < |\phi(u'g')|$ and $|\phi(u'')| \le |s''| < |\phi(g''u'')|$. Define $t', t'' \in \Sigma^*$ by $s' = \phi(u')t'$ and $s'' = t''\phi(u'')$. Then

$$\phi(u) = \phi(u')\phi(g')\phi(g'')\phi(u'') = \phi(u')t'\ell t''\phi(u''),$$

so $\phi(g')\phi(g'') = t'\ell t''$. By $|s''| < |\phi(g''u'')|$ we get $|t''| < |\phi(g'')|$. Define $\ell'' \in \Sigma^+$ by $\phi(g'') = \ell'' t''$. Define $\ell' \in \Sigma^*$ by $\ell = \ell'\ell''$. So $\phi(g') = t'\ell'$. By $|s'| < |\phi(u'g')|$ we get $|t'| < |\phi(g')|$ and so $\ell' \in \Sigma^+$.

Since $\phi(g'')$ is a prefix of r, we obtain $\ell'' \in \mathrm{OVL}(\ell, r)$, so $\ell'' = \beta$ and $\ell' = \ell_\beta$. Define $h', h'' \in \Omega$ by

$$h' = \begin{cases} d & \text{if } g' = a, \\ e & \text{if } g' = c, \end{cases} \qquad h'' = \begin{cases} c & \text{if } g'' = a, \\ e & \text{if } g'' = d, \\ b & \text{if } g'' = f. \end{cases}$$

Then $g'g'' \to h'fh''$ is in R, and moreover $\phi(g') = \phi(h')\ell_\beta = t'\ell_\beta$ and $\phi(g'') = \beta\phi(h'') = \beta t''$. So $t' = \phi(h')$ and $t'' = \phi(h'')$ and so

$$t = s' r s'' = \phi(u')\phi(h')\phi(f)\phi(h'')\phi(u'') = \phi(v)$$

for $v = u'h'fh''u''$. So $u \to_R v$. By Lemma 6.3 we get $v \in \mathcal{M}$. □

We are about to prove termination of $\ell \to r$ by a reduction to termination of R. For this purpose we still need $\{\ell \to r\}$-reductions that start in $\phi[\mathcal{M}]$. Such reductions are provided by forward closures [10, 1] as we will show next. We use the following characterization of forward closures by Hermann.

DEFINITION 6.6 ([6, Corollaire 2.16]). *The set of forward closures of a string rewriting rule $\ell \to r$ over alphabet Σ is the least set $\mathrm{FC}(\ell \to r)$ of $\ell \to r$-reductions such that*

fc1. $(\ell \to r) \in \mathrm{FC}(\ell \to r)$,

fc2. if $(s_1 \to^+ t_1'\ell') \in \mathrm{FC}(\ell \to r)$ *and* $\ell = \ell'\ell''$ *for some* $\ell', \ell'' \in \Sigma^+$ *then* $(s_1\ell'' \to^+ t_1'\ell'\ell'' \to^+ t_1'r) \in \mathrm{FC}(\ell \to r)$,

fc3. if $(s_1 \to^+ t_1'\ell t_1'') \in \mathrm{FC}(\ell \to r)$ *then* $(s_1 \to^+ t_1'\ell t_1'' \to^+ t_1' r t_1'') \in \mathrm{FC}(\ell \to r)$.

LEMMA 6.7. *Every forward closure of a rule $\ell \to r$ of the form* (4.3) *where $\ell_\beta\ell_\beta$ is not a suffix of r_α, has a right hand side in $\phi[\mathcal{M}]$.*

Proof. By induction along the definition of forward closure. Let $(s \to^+ t) \in \mathrm{FC}(\ell \to r)$. In Case (fc1) we have $t = r = \phi(f)$. In Case (fc3) the claim follows from Lemma 6.5. This leaves to prove Case (fc2).

Suppose that $s = s_1\ell''$, $t = t_1'r$, $(s_1 \to^+ t_1'\ell') \in \mathrm{FC}(\ell \to r)$, and $\ell = \ell'\ell''$ for some $\ell', \ell'' \in \Sigma^+$. By inductive hypothesis, there is $u \in \mathcal{M}$ such that $t_1'\ell' = \phi(u)$. By definition of $\mathcal{M}$, u has suffix f or fb.

Case 1: u has suffix fb. Define $g' \in \Omega^*$ by $u = g'fb$. Then

$$g' \in (a + d(fe)^* + d(fe)^*fc)^*d(fe)^*$$

by definition of $\mathcal{M}$. We distinguish cases whether $|\ell'| > |r_\beta|$ or not.

Case 1.1: $|\ell'| > |r_\beta|$. The string $t_1'\ell'$ has suffix $\phi(fb) = rr_\beta$. By $|\ell| < |r|$ and $|\ell'| > |r_\beta|$ we get $\ell' = zr_\beta$ for some non-empty suffix z of r. Now $z \in \mathrm{OVL}(r, \ell)$, so $z = \alpha$. So $t_1'\ell' = \phi(g')rr_\beta = \phi(g')r_\alpha\ell'$, whence $t_1' = \phi(g')r_\alpha = \phi(g'a)$. So $t_1'r = \phi(g'a)r = \phi(g'af)$ for $g'af \in \mathcal{M}$.

Case 1.2: $|\ell'| \le |r_\beta|$. Then ℓ' is a suffix of r_β and so of r. So $\ell' \in \mathrm{OVL}(r, \ell)$ whence $\ell' = \alpha$. So $t_1'\ell' = \phi(g'f)r_\beta = \phi(g'f)r_{\beta,\alpha}\ell'$, whence $t_1' = \phi(g'f)r_{\beta,\alpha} = \phi(g'fc)$. So $t_1'r = \phi(g'fc)r = \phi(g'fcf)$ for $g'fcf \in \mathcal{M}$.

Case 2: u has suffix f. Define $g' \in \Omega^*$ by $u = g'f$. Then

$$g' \in (a + d(fe)^* + d(fe)^*fc)^*$$

by definition of $\mathcal{M}$. By $|\ell| < |r|$ we get that $\ell' \in \mathrm{OVL}(r, \ell)$, whence $\ell' = \alpha$. So $t_1'\ell' = \phi(g'f) = \phi(g')r = \phi(g')r_\alpha\ell'$, whence $t_1' = \phi(g')r_\alpha = \phi(g'a)$. So $t_1'r = \phi(g'a)r = \phi(g'af)$ for $g'af \in \mathcal{M}$. □

LEMMA 6.8. *A rule $\ell \to r$ of the form* (4.3) *terminates if $\ell_\beta\ell_\beta$ is not a suffix of r_α.*

Proof. If $\ell \to r$ is non-terminating then there is an infinite rewriting sequence $s_1 \to_{\ell\to r} s_2 \to_{\ell \to r} \cdots$ starting from a right hand side of a forward closure [1]. By Lemma 6.7 $s_1 \in \phi[\mathcal{M}]$, i.e., there is $u_1 \in \mathcal{M}$ such that $\phi(u_1) = s_1$. By induction on i, using Lemma 6.5, one easily proves that for every i there is an $u_{i+1} \in \mathcal{M}$ such that both $u_i \to_R u_{i+1}$ and $\phi(u_{i+1}) = s_{i+1}$. Hence we get an infinite reduction sequence $u_1 \to_R u_2 \to_R \ldots$. Contradiction to termination of R. □

EXAMPLE 2. *For every $m \ge 0$, the one-rule SRS*

$$ab(dab)^{m+1}ab \to dababb(dab)^{m+1}a$$

is terminating by Lemma 6.8. With $m = 0$ we get the smallest terminating witness ($|r| = 10$) of Lemma 4.4.

This example also proves that Kurth's [8] Criterion F is incomplete, for Criterion F applies only to the left barren or right barren cases [3, Theorem 6.31].

We note moreover that the maximal length of a derivation starting with $s \in \Sigma^*$ is linear in $|s|$. This is a direct consequence of the decreasing weight associated with a step $u \to_R v$.

7. The Main Theorem. Now we have all material together to prove our claim.

THEOREM 7.1. *Let* $|\operatorname{OVL}(r,\ell)| = |\operatorname{OVL}(\ell,r)| = 1$. *Then* $\{\ell \to r\}$ *terminates if and only if it has no loop of lengths 1, 2, or 3.*

Proof. Let $\operatorname{OVL}(r,\ell) = \{\alpha\}$ and $\operatorname{OVL}(\ell,r) = \{\beta\}$. If ℓ is a factor of r then $\{\ell \to r\}$ has a loop of length 1 [8]. Else if $|\ell| \geq |r|$ then $\{\ell \to r\}$ terminates. If $\ell \to r$ is left barren or right barren then $\{\ell \to r\}$ terminates. So suppose that ℓ is not a factor of r; that $|\ell| < |r|$; and that $\ell \to r$ is neither left barren nor right barren. We distinguish cases:

Case 1: $\ell \to r$ is neither left s-barren nor right s-barren. Then $r = r'\ell_\beta\alpha$ and $r = \beta\ell_\alpha r''$ for some strings r', r''. There is a loop of length 2:

$$\ell\ell_\alpha \to r\ell_\alpha = r'\ell_\beta\alpha\ell_\alpha = r'\ell_\beta\ell \to r'\ell_\beta r = r'\ell_\beta\beta\ell_\alpha r'' = r'\ell\ell_\alpha r''.$$

Case 2: $\ell \to r$ is left s-barren but not right s-barren. Then $\ell \to r$ has the form (4.3). If $\ell_\beta\ell_\beta$ is a suffix of r_α then $\{\ell \to r\}$ has a loop of length 3 by Lemma 5.1. Else $\{\ell \to r\}$ terminates by Lemma 6.8.

Case 3: $\ell \to r$ is not left s-barren but right s-barren. This case is symmetric to Case 2: We have a loop of length 3 if $\ell_\alpha\ell_\alpha$ is a prefix of r_β, otherwise termination.

Case 4: $\ell \to r$ is both left s-barren and right s-barren. Then Lemma 4.1 and its dual apply, showing $|\beta| > |\alpha|$ and $|\alpha| > |\beta|$, a contradiction. So this case does not exist. This finishes the proof. ∎

Kurth [9] has proved decidability of the existence of loops of lengths 1, 2, or 3 for one-rule SRSs. Indeed, for every SRS and every $n \geq 1$, the existence of loops of lengths less or equal n is decidable [5].

COROLLARY 7.2. *Termination is decidable for one-rule SRSs* $\{\ell \to r\}$ *that satisfy* $|\operatorname{OVL}(r,\ell)| = |\operatorname{OVL}(\ell,r)| = 1$.

8. Conclusion. We proved that termination of one-rule SRSs with one pair of overlaps is equivalent to the non-existence of loops of length less than or equal to 3. Thus we showed that termination is decidable for one-rule SRSs with one pair of overlaps. A surprising observation in this investigation was the emergence of non-tame rules, some admitting loops of length 3, and some terminating. Such rules were not covered by the two precursor results by Kurth and by Shikishima-Tsuji et al.

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This paper presents a partial solution to the long standing open problem of termination of one-rule string rewriting. Overlaps between the two sides of the rule play a central role in existing termination criteria. We characterize termination of all one-rule string rewriting systems that have one such overlap at either end. This both completes a result of Kurth and generalizes a result of Shikishima-Tsuji et al.

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