A Meshless Local Petrov-Galerkin Method for Euler-Bernoulli Beam Problems

I. S. Raju[1] and D. R. Phillips[2]
NASA Langley Research Center, Hampton, Virginia

Summary

An accurate and yet simple Meshless Local Petrov-Galerkin (MLPG) formulation for analyzing beam problems is presented. In the formulation, simple weight functions are chosen as test functions. The use of these functions shows that the weak form can be integrated with conventional Gaussian integration. The MLPG method was evaluated by applying the formulation to a variety of patch test and thin beam problems. The formulation successfully reproduced exact solutions to machine accuracy when test functions with C^2 continuity and an appropriate order of basis functions are used.

Introduction

Meshless methods are increasingly being viewed as an alternative to the finite element method [1-3]. Recently, a meshless Galerkin formulation was presented for beam (C^1) problems using a generalized moving least squares (MLS) interpolant [2]. In this Galerkin formulation, the derivatives of the weight functions used in the trial and test functions higher than the second order showed discontinuities (scissors) at the boundaries of the supports of the local sub-domains. Conventional Gaussian and other integration schemes were found to be inadequate to integrate the weak form accurately – very accurate and elaborate integration schemes were needed.

In this paper an accurate and simple alternative Petrov-Galerkin formulation for beam problems is presented. In the present Petrov-Galerkin method simple weight functions are chosen as test functions and conventional Gaussian integration is used. The effectiveness of the MLPG method is evaluated by applying the formulation to a variety of patch test problems.

Development of the Petrov-Galerkin Formulation

The notation of reference 2 is used in this paper for brevity and convenience in presentation. The MLPG equations are

$$\mathbf{K}_i^{(\text{node})}\mathbf{d} + \mathbf{K}_i^{(\text{bdy})}\mathbf{d} - \mathbf{f}_i^{(\text{node})} - \mathbf{f}_i^{(\text{bdy})} = \mathbf{0} \tag{1}$$

where

$$\{\mathbf{d}\} = \left\{ \hat{u}^1, \quad \hat{q}^1, \quad \hat{u}^2, \quad \hat{q}^2, \quad \ldots \right\}^{\mathrm{T}} \tag{2}$$

are the fictitious nodal values of deflections u and slopes q, and the matrices in Eq. (1) are defined as in Eq. 35-36(g) of reference 2.

---

[1] Head, Analytical and Computational Methods Branch, NASA Langley Research Center

[2] Joint Institute for Advancement of Flight Sciences – George Washington University, performed under grant NCC1-384.

The MLPG equations are derived using a weighted residual weak form of the governing equations. The trial functions used for beam problems are derived using the generalized MLS interpolation [2] as

$$u(x) = \sum_{i=1}^{N} \left(\hat{u}^i y_i^{(u)}(x) + \hat{q}^i y_i^{(q)}(x) \right) \tag{3}$$

where

$$\begin{aligned} y_i^{(u)}(x) &= \sum_{j=1}^{m} p_j(x) \left[\mathbf{A}^{-1} \mathbf{P}^{\mathrm{T}} \mathbf{w} \right]_{ji} \\ y_i^{(q)}(x) &= \sum_{j=1}^{m} p_j(x) \left[\mathbf{A}^{-1} \mathbf{P}_x^{\mathrm{T}} \mathbf{w} \right]_{ji} \end{aligned} \tag{4}$$

with

$$[\mathbf{A}] = \mathbf{P}^{\mathrm{T}} \mathbf{w} \mathbf{P} + \mathbf{P}_x^{\mathrm{T}} \mathbf{w} \mathbf{P}_x \quad . \tag{5}$$

In Eq. (5) **P** is an (n,m) matrix and **w** is an (n,n) matrix defined as

$$[\mathbf{P}] = \left[\, \mathbf{p}(x_1) \quad \mathbf{p}(x_2) \quad \ldots \quad \mathbf{p}(x_n) \, \right]^{\mathrm{T}} \quad , \tag{6}$$

$$\mathbf{w} = \begin{bmatrix} w_1(\bar{x}) & & & \\ & w_2(\bar{x}) & & \\ & & \ddots & \\ & & & w_n(\bar{x}) \end{bmatrix} \tag{7}$$

where $\bar{x} = x - x_i$, and

$$\mathbf{p}^{\mathrm{T}}(x) = \left[\, 1, \quad x, \quad x^2, \quad \ldots \quad x^{m-1} \, \right] \quad , \tag{8a}$$

$$\mathbf{p}_x^{\mathrm{T}}(x) = \frac{d\mathbf{p}^{\mathrm{T}}(x)}{dx} = \left[\, 0, \quad 1, \quad 2x, \quad \ldots \quad (m-1)x^{m-2} \, \right] \tag{8b}$$

with (m-1) as the order of the basis function **p**(x) used in the MLS approximation. Three different weight functions $w_i(\bar{x})$ are considered:

$$w_i(\bar{x}) = \begin{cases} \left[\, 1 - \left\| (x - x_i)^2 \right\| / R_i^2 \, \right]^a & \text{if} \quad \left\| x - x_i \right\| \le R_i \\ 0 & \text{if} \quad \left\| x - x_i \right\| > R_i \end{cases} \tag{9}$$

with a = 2, 3, and 4. Note that the extent of the influence of the trial function is controlled by distance R_i.

Several different test functions $v_i(x)$ are considered:

$$v_i(x) = \begin{cases} \left[1 - \left\| (x - x_i)^2 \right\| / R_o^2 \right]^b & \text{if} \quad \|x - x_i\| \le R_o \\ 0 & \text{if} \quad \|x - x_i\| > R_o \end{cases} \tag{10}$$

with $ß$ = 2, 3, and 4,

$$v_i(x) = \begin{cases} 1 - 3\left(\frac{d_i}{R_o}\right)^2 + 2\left(\frac{d_i}{R_o}\right)^3 & \text{if} \quad 0 \le d_i \le R_o \\ 0 & \text{if} \quad d_i \ge R_o \end{cases} \tag{11}$$

with $d_i = \| x - x_i \|$, and

$$v_i(x) = \begin{cases} 1 - 6\left(\frac{d_i}{R_o}\right)^2 + 8\left(\frac{d_i}{R_o}\right)^3 - 3\left(\frac{d_i}{R_o}\right)^4 & \text{if} \quad 0 \le d_i \le R_o \\ 0 & \text{if} \quad d_i \ge R_o \end{cases} \quad . \tag{12}$$

The derivatives of the test functions are evaluated at the center (d_i /R_o = 0) and at the end points (d_i /R_o = 1) as

$$\frac{\partial^{m_0}}{\partial x^{m_o}} v_i\left(\frac{d_i}{R_o} = 0\right) = 0 \quad ; \quad m_0 \ge 1 \qquad \text{and} \qquad \frac{\partial^{m_1}}{\partial x^{m_1}} v_i\left(\frac{d_i}{R_o} = 1\right) = 0 \quad ; \quad m_1 \ge 1 \tag{13}$$

The test functions are then C^γ continuous up to the order γ where $\gamma = \min(m_0, m_1)$ (See reference 4, pp. 63-64). With these definitions, the test functions in Eq. (10) with $ß$ = 2, 3, and 4 are C^1, C^1, and C^3 continuous, respectively. Similarly, the spline functions in Eqs. (11) and (12) are C^1 continuous. Note that the lengths R_i and R_o in Eqs. 9-12 are user defined in the MLPG method.

As the test functions chosen (Eqs. 10-12) are not the same as the trial functions, the current implementation is a Petrov-Galerkin method. This is in contrast to reference 2 where a Galerkin method was used. The Galerkin method led to discontinuities (scissors) at the boundaries of the supports of the trial functions. Due to these scissors, elaborate numerical integration schemes were needed to integrate the weak form accurately. The domain of dependence Ω_s was subdivided into subregions depending on the ends of the support domains of various trial functions. A 10-point Gaussian quadrature was used in each of these subregions to integrate the weak from accurately. In the current implementation with the Petrov-Galerkin test functions, the integrands were examined and no discontinuities were found in the domain of integration. A single domain for the entire compact support domain (Ω_s) was used. A standard 8-point Gaussian quadrature rule was found to be sufficient to integrate the weak form very accurately.

Beam Configurations and Models

A beam of constant flexural rigidity EI and a length of $4l$ is considered. Six models with 5, 9, 17, 33, 65, and 129 nodes uniformly distributed along the length of the beam are considered. Figure 1 shows a typical 17-node model. The distances between the nodes (Δ/l) in these models are 1.0, 0.5, 0.25, 0.125, 0.0625, and 0.03125, respectively for the 5-, 9-, 17-, 33-, 65-, and 129-node models. Three types of basis functions, quadratic basis ($1, x, x^2$), cubic basis ($1, x, x^2, x^3$), and quartic basis ($1, x, x^2, x^3, x^4$) are used. The local coordinate approach of reference 5 is used in evaluating the shape functions and their derivatives. System matrices in Eq. (1) are developed with these parameters.

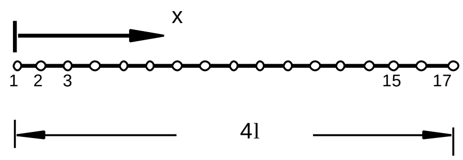


Figure 1: A 17-node model of the beam

Numerical Evaluations

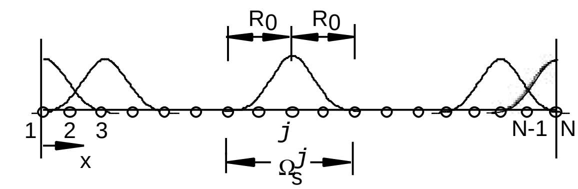
The current MLPG formulation was evaluated by applying the formulation to simple patch-test problems. Three typical problems were considered.

$$\begin{aligned}
&1.\quad u(x) = c_0, \qquad q = \frac{du}{dx} = 0 \quad ; \quad \text{Rigid body translation} \\
&2.\quad u(x) = c_1 x, \qquad q = c_1 \quad ; \quad \text{Rigid body rotation} \\
&3.\quad u(x) = c_2 x^2, \qquad q = 2c_2 x \quad ; \quad \text{Constant - curvature condition}
\end{aligned} \tag{14}$$

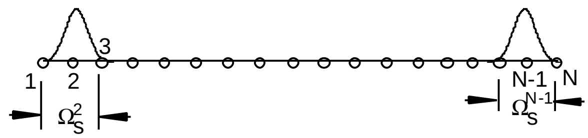
All three of these problems satisfy the governing differential equation exactly and as such are exact solutions. The deflection u and the slope q corresponding to problems 1, 2, and 3 were prescribed as essential boundary conditions (EBCs) at $x=0$ and $x=4l$. With these EBCs, the beam problem was solved using the MLPG method in Eq. (1). When ($R_o / l = 2\Delta$) and ($R_i / l = 8\Delta$) were used, each of the 5-, 9-, 17-, 33-, 65-, and 129-node models reproduced exact solutions at all internal nodes to machine accuracy thus "passing" the patch tests. The analysis was repeated with each of the test functions in Eqs. (10-12), and the MLPG algorithm passed the patch tests for all cases.

As mentioned previously, the parameters (R_o / l) and (R_i / l) in the MLPG method are user-controlled. The previously mentioned lengths ($R_o / l = 2\Delta$) and ($R_i / l = 8\Delta$) were used at all nodes of an N-node model, except at node 2 and node N-1 (see Figure 2). For these nodes ($R_o / l = \Delta$) was used to ensure a symmetric Ω_s. Note that with these assignments of (R_o / l) the test functions for all interior nodes have symmetric Ω_s configurations. As shown in Figure 2, no asymmetry is introduced at nodes 1 and N as exactly half of their test functions are used. When these symmetries were violated, the MLPG method failed the patch tests.

As the models are refined, the value of (Δ/l) decreases and thus the size of Ω_s and the extent of the trial functions also decrease. For finer models, i.e. for the 33-, 65-, and 129-node models, when $8\Delta \le R_i / l \le 16\Delta$ the MLPG method yielded very accurate results. However, when $R_i / l > 16\Delta$ the MLPG method failed the patch tests for these models. When $R_i / l > 16\Delta$ the trial function is too diffused and the size of Ω_s ($R_o / l = 2\Delta$) is too small in comparison to (R_i / l). The combination of small Ω_s size and large (R_i / l) are apparently incompatible. While the finer models performed well over a large range of (R_i / l), the coarser models performed well in a much smaller range of (R_i / l). For good performance, (R_i / l) needed to be around 8Δ but less than 98% of the total beam length.



(a) Ω_S for nodes the 1st, jth and Nth nodes



(b) Ω_S for nodes the 2nd and (N-1)st nodes

Figure 2: Ω_S definitions for various nodes

The MLPG method was applied to beam problems with mixed boundary conditions. The first problem considered was a cantilever beam with a concentrated moment at the free end (i.e. $M = M_0$ at $x = 4l$). The exact solution for this problem is $u = M_0x^2 / 2EI$ and $q = M_0x / EI$. For all trail functions considered, the MLPG algorithm reproduced the exact solution when the test functions in Eq. (10) with ß = 3 and 4 and when the 4-term spline function in Eq. (12) were used. In contrast, the algorithm failed to reproduce the exact solution when the test function in Eq. (10) with ß = 2 and the 3-term spline function of Eq. (11) were used. This example suggests that test functions with at least C^1 continuity and with $m_1 \geq 2$ (see Eq. 13) are required for the MLPG algorithm for beam problems.

The second problem considered was a cantilever beam with a tip load. Since the exact solution for this problem is a cubic in terms of the x-coordinate of the beam, all six models with cubic basis functions and a test function with C^1 continuity and with $m_1 \geq 2$ reproduced the exact solution to machine accuracy.

The third problem considered was a simply supported beam subjected to a uniformly distributed load. Using symmetry, half of the beam was modeled. Since the exact solution for this problem is quartic in terms of the x-coordinate of the beam, the MLPG method with a cubic basis function did not reproduce the exact solution. Error norms defined as

$$\|E_u\|_2 = \sqrt{\frac{1}{g}\sum_{k=1}^{g}\left[\frac{(u_{MLPG} - u_{exact})}{u_{exact}}\right]_k^2}\ ;\ \ \|E_M\|_2 = \sqrt{\frac{1}{g}\sum_{k=1}^{g}\left[\frac{(M_{MLPG} - M_{exact})}{M_{exact}}\right]_k^2} \tag{15}$$

were computed at g uniformly spaced points along the beam. A value of $g = 200$ was used. The norms $\|E_u\|_2$ and $\|E_M\|_2$ are presented in Table 1. The numbers in Table 1 were computed using a 20-point Gaussian in each of the single compact support domains Ω_s. As expected, all models yielded accurate solutions (within 4% for u, q, and M). When the order of the basis function was increased to quartic, the MLPG method reproduced the exact solutions (for u, q, M, and V) to machine accuracy.

Table 1: Error norm $||E||_2$ for a simply supported beam subjected to a uniformly distributed load with cubic basis used in the MLPG method. (Trial function using Eq. (9) with a=3 and test function of Eq.(10) with β=4.)

Error norm	Number of nodes in the model					
	5*	9*	17	33	65	129
$\|\|E_u\|\|_2$	0.1662e-1	0.1306e-2	0.4573e-2	0.3829e-1	0.1742e-1	0.2368e-1
$\|\|E_M\|\|_2$	0.2774e+0	0.1057e-1	0.1704e-1	0.3680e-1	0.1763e-1	0.2340e-1

* R_i / l = 3.9

The above problem demonstrates the following phenomena. When the order of the basis function equals the order of the exact solution, the previously stated 8-point Gaussian in a single Ω_s is sufficient to integrate the weak form very accurately. When the order of the basis function is less than the order of the exact solution, a higher order integration rule (such as a 20-point Gaussian) is needed to obtain accurate results. Note that a 20-point Gaussian in a domain Ω_s is preferable over the sub-domain integration required in the Galerkin formulation.

Concluding Remarks

This paper presents an accurate and yet a simple Petrov-Galerkin formulation for solving beam problems. In the formulation, simple weight functions are chosen as test functions. These test functions must have at least C^1 continuity and zero first and second derivatives at the end points for the algorithm to pass the patch tests. When appropriate test functions are used, the weak form can be integrated with conventional Gaussian integration. The MLPG method was evaluated by applying the formulation to a variety of patch tests. Weight functions used to develop the trial functions need not have C^1 continuity; C^0 continuity may be sufficient. Additionally, the weak form requires at least a quadratic basis, and a cubic basis function is preferable. The formulation successfully reproduced exact solutions to machine accuracy for thin beam problems when an appropriate order of basis function was used for all combinations of trial and C^1 continuous test functions considered.

References

[1] Atluri, S. N. and Zhu, T. (1998): "A new Meshless Local Petrov-Galerkin (MLPG) approach in computational mechanics," *Computational Mechanics*, Vol. 22, pp. 117-127.

[2] Atluri, S. N., Cho, J. Y., and Kim, H. -G. (1999): "Analysis of thin beams, using the meshless local Petrov-Galerkin method, with generalized moving least squares interpolations," *Computational Mechanics*, Vol. 24, pp. 334-347.

[3] Gu, Y. T. and Liu, G. R. (2001): "A local point interpolation method for static and dynamic analysis of thin beams," *Computer Methods in Applied Mechanics and Engineering*, Vol. 190, pp. 5515-5528.

[4] Atluri, S. N. and Shen, S. (2002): *The Meshless Local Petrov-Galerkin (MLPG) Method*, Tech Science Press, Encino, CA.

[5] Raju, I. S. and Phillips, D. R. (2002): "A Local Coordinate Approach in the MLPG Method for Beam Problems," NASA TM-2002-211463.