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A numerical scheme for ordinary differential equations having time varying and nonlinear coefficients based on the state transition matrix

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Abstract

A variable order method of integrating initial value ordinary differential equations that is based on the state transition matrix has been developed. The method has been evaluated for linear time variant and nonlinear systems of equations. While it is more complex than most other methods, it produces exact solutions at arbitrary time step size when the time variation of the system can be modeled exactly by a polynomial. Solutions to several nonlinear problems exhibiting chaotic behavior have been computed. Accuracy of the method has been demonstrated by comparison with an exact solution and with solutions obtained by established methods.

Section 1.0. Introduction. Numerical methods to solve ordinary differential equations are at a state where efficient and accurate solutions are routine. Accuracy is often achieved by adapting time step size, which in some applications is inconvenient or impossible. It may not be possible to adapt time step size when the equations of dynamics are integrated while coupled to another solver (e.g. computational fluid dynamics (CFD) code). In other cases an arbitrarily large constant time step size may be desirable such as when externally supplied data is applied to a system at constant time intervals.

One approach to address this dilemma has been the construction of numerical schemes based on classical solutions. An exact numerical scheme based on the state transition matrix (STM) is an example. One advantage of a scheme based on the STM is that it offers an exact solution of the system for arbitrary time step size. The ability to compute at arbitrary time step size is useful, for instance, when coupling a continuous system to a discretely sampled control system with discrete feed back. The system can be integrated with absolute accuracy between samples and discrete feed back with one or several steps, as required to capture the discrete nature of the problem. While numerical methods for time invariant systems based on the STM have been in use for many years[1], only recently has it been applied to time variant systems.[2] The present paper proposes a discrete form of the STM that has application to any linear time varying (LTV) system, and some nonlinear time varying (NLTV) systems.

Consider the simple first order differential equation

$$\dot{x} = A(t)x + B(t)u(t) \tag{1}$$

where $x = (x_1,...,x_N)^T$, $x_n \in R$, $A \in R^{N\times N}$, $B \in R^{N\times N}$ $t \geq t_0$. Setting the dimension of the matrix B identical to matrix A is purely for convenience; the results are applicable to a matrix B of general form. In this paper the classical method of solving differential equations will be used to construct the numerical solution of equation (1). Between times

t_m and $t_m + h$, where h is time step size, a discrete approximation will be performed using the solution

$$x(t_m + h) = \Phi(t_m + h, t_m)x(t_m) + \int_{t_m}^{t_m+h} \Phi(t_m + h, T)B(T)u(T)dT \tag{2}$$

For any LTV system, the STM between times t_m and $t_m + h$ is given by the Neumann series

$$\begin{aligned}\Phi(t_m + h, t_m) = I &+ \int_{t_m}^{t_m+h} A(t)dt + \int_{t_m}^{t_m+h} A(t_1)\int_{t_m}^{t_1} A(t_2)dt_2 dt_1 \\ &+ \int_{t_m}^{t_m+h} A(t_1)\int_{t_m}^{t_1} A(t_2)\int_{t_m}^{t_2} A(t_3)dt_3 dt_2 dt_1 + \cdots\end{aligned} \tag{3}$$

This result is called the Peano-Baker formula. Only in special cases, such as when the matrix A is constant, is this solution equal to the more familiar expression

$$\Phi(t_m + h, t_m) = \exp\left[\int_t^{t+h} A\, dt\right] = I + Ah + \tfrac{1}{2}(Ah)^2 + \cdots.$$

In all applications discussed in this paper, a discrete form of equation (3) will be used. That solution may have some value if an approximation can be found that is efficient, accurate and robust.

Section 2.0. Approximation of the *A(t)* matrix for an LTV system. Consider the discrete analog to the continuous functions above.

$$x_n^{-1} = x_n(t_m - h) \quad , \quad x_n^0 = x_n(t_m) \quad , \quad x_n^1 = x_n(t_m + h) \quad , \ldots, \quad x_n^r = x_n(t_m + rh)$$

$$a_{n,l}^{-1} = a_{n,l}(t_m - h) \quad , \quad a_{n,l}^0 = a_{n,l}(t_m) \quad , \quad a_{n,l}^1 = a_{n,l}(t_m + h) \quad , \ldots, \quad a_{n,l}^r = a_{n,l}(t_m + rh)$$

$$b_{n,l}^{-1} = b_{n,l}(t_m - h) \quad , \quad b_{n,l}^0 = b_{n,l}(t_m) \quad , \quad b_{n,l}^1 = b_{n,l}(t_m + h) \quad , \ldots, \quad b_{n,l}^r = b_{n,l}(t_m + rh)$$

$$u_n^{-1} = u_n(t_m - h) \quad , \quad u_n^0 = u_n(t_m) \quad , \quad u_n^1 = u_n(t_m + h) \quad , \ldots, \quad u_n^r = u_n(t_m + rh)$$

The numerical solution of equation (1) can be obtained by starting with a term by term polynomial approximation of the matrix *A(t)*

$$a_{n,l}(\tau) = \sum_{i=1}^{I} c_{i,n,l}\tau^{i-1} / h^{i-1} \quad , \quad \tau = t - t_m \ , \ \tau \in (0, h) \tag{4}$$

for schemes of various order. For instance, one can define the O(Δ) scheme where $\Delta = I - 1$. The coefficients in equation (4) are constructed from discrete values of the terms of matrix *A(t)*, that is $c_{i\,n,l} = c_{i\,n,l}(a^{1}_{n,l},\ldots,a^{-I}_{n,l})$. These interpolation polynomials can be easily evaluated since $A(t)$ is known for time steps at least up to t_m+h. To illustrate the use of equation (4), the $a_{n,l}$ term of the continuous matrix *A(t)* can be approximated by polynomial expressions of orders zero, one and two for $\tau \in (0,h)$ by the following

O(0) scheme: $a_{n,l}(\tau) = c_{1\,n,l}$ where $c_{1\,n,l} = a^{0}_{n,l}$

O(I) scheme: $a_{n,l}(\tau) = c_{1\,n,l} + c_{2\,n,l}\tau / h$ where $c_{1\,n,l} = a^{0}_{n,l}$, $c_{2\,n,l} = a^{1}_{n,l} - a^{0}_{n,l}$

O(II) scheme: $a_{n,l}(\tau) = c_{1\,n,l} + c_{2\,n,l}\tau / h + c_{3\,n,l}\tau^2 / h^2$

where

$$c_{1\,n,l} = a^{0}_{n,l} \quad , \quad c_{2\,n,l} = \tfrac{1}{2}\left(a^{1}_{n,l} - a^{-1}_{n,l}\right) \quad , \quad c_{3\,n,l} = \tfrac{1}{2}\left(a^{1}_{n,l} - 2a^{0}_{n,l} + a^{-1}_{n,l}\right)$$

In a similar manner, higher order approximations of the matrix *A(t)* can be constructed term by term. The accuracy of this scheme is dependent on the accuracy of the approximation of matrix *A(t)* since equation (3) is exact. Under certain circumstances a method based on equation (3) will be an exact numerical scheme.

Section 2.1. Approximation of the STM for an LTV system. Using the results of section 2.0, an algorithm is constructed to compute successive terms of the Peano-Baker formula. Since polynomials of order $\Delta = I - 1$ are used to approximate the matrix *A(t)* term by term, the integration of those expressions over $(t_m, t_m + h)$ can be accomplished in a straight forward manner. The first two terms are easily found. The computation begins as follows:

$$(\sigma^{i}_{n,l})_1 = c_{i,\,n,l} / i \quad , \quad i = 1, I \quad , \quad n = 1, N \ , \quad l = 1, N \qquad (5\,a)$$

$$\Phi_{n,l}(t_m + h, t_m) = \delta_{n,l} + h\sum_{i=1}^{I}(\sigma^{i}_{n,l})_1 \quad , \quad n = 1, N \ , \quad l = 1, N \qquad (5\,b)$$

where $\delta_{n,l} = 1$ if $n = l$ and $\delta_{n,l} = 0$ otherwise. The superscript *i* in equations (5a-b) is an index ranging from 1 to I. In equation (4) the same index *i* is used as an exponent. Equations (5 a) and (5 b) compute the sum of the first two terms in the series equation (3) and provide the start of the STM. The terms $(\sigma^{i}_{n,l})_1$ computed in equation (5a) are the kernel of a recursion that computes the remaining terms of the STM.

Starting at the double integral (the third term in equation (3)) the following recursive algorithm computes the remaining integral terms and adds their contribution to the STM.

For $j = 0$, J

$$\left.\begin{aligned} (\sigma_{n,l}^{p})_2 &= \sum_{k=1}^{p}\sum_{i=1}^{N} c_{k,n,i}(\sigma_{i,l}^{p-k+1})_1/(p+j+1) \qquad (\forall\, I) \\ (\sigma_{n,l}^{-p+j(I-1)+2I})_2 &= \sum_{k=1}^{p}\sum_{i=1}^{N} \frac{c_{-k+I+1,n,i}(\sigma_{i,l}^{-p+k+j(I-1)+I})_1}{-p+I(j+2)+1} \\ & \qquad\qquad (I>1) \end{aligned}\right\} \begin{aligned} & p = 1, I-1 \quad , \\ & n = 1, N\ , \quad l = 1, N \end{aligned} \qquad \begin{aligned} &(6\,a) \\ &(6\,b) \end{aligned}$$

$$(\sigma_{n,l}^{p+1})_2 = \sum_{k=1}^{I}\sum_{i=1}^{N} c_{k,n,i}(\sigma_{i,l}^{p-k+2})_1/(p+j+2) \qquad (I>1) \qquad (6\,c)$$

$$,\ \ p = I-1, (j+1)(I-1), \quad n = 1, N\ , \quad l = 1, N$$

$$\Phi_{n,l}(t_m+h, t_m) = \Phi_{n,l}(t_m+h, t_m) + h^{j+2}\sum_{k=1}^{2I-1+j(I-1)}(\sigma_{n,l}^{k})_2 \quad , \quad n = 1, N\ , \quad l = 1, N \qquad (6\,d)$$

$$(\sigma_{n,l}^{k})_1 = (\sigma_{n,l}^{k})_2 \qquad k = 1, 2I-1+j(I-1) \quad , \quad n = 1, N\ , \quad l = 1, N \qquad (6\,e)$$

end j

Successive terms in the integral series are computed starting at $j = 0$ with the third term (second integral). When $j = 0$, the outer subscript $(\ )_1$ denotes values computed in equations (5). When $j > 0$, the outer subscript $(\ )_1$ denotes values from the previous iteration of equations (6). There are a total of $J + 3$ terms of the Neumann series computed by this algorithm, combined from both sets of formulae, equations (5) and (6). For instance if $J = 0$, the first 3 terms are computed, namely the first two obtained from the former equations and that with $j = 0$ from the latter recursive formulae. When implemented, several indexing details are important. If $I = 1$, the index p ranging over 1 to $(I-1)$ is computed once for $p = 1$, and the sequence with index p over $(I-1)$ to $(j+1)(I-1)$ is computed once for $p = 0$.

When $I = 1$, this solution is just the exponential series for a constant matrix (i.e. *exp(Ah)*) or alternately a zero order approximation of a time varying *A(t)* matrix. Solution of an LTV system at this level of approximation corresponds simply to applying the exponential series of the steady state STM updated at each time step. Furthermore, when the O(0) scheme is used ($I = 1$) and $J = -1$ (i.e. only the first two terms computed), this algorithm is the explicit backward Euler method. Since this scheme uses as its basis an exact solution, it possible with enough terms to compute solutions using large time step sizes, as long as the matrix *A(t)* is sufficiently well behaved so as to be accurately approximated by a polynomial expansion. In the cases computed so far, relatively few terms in the series have been required to achieve very good accuracy.

It would be desirable to obtain a bound on the size of the leading order truncated term following the last of the $J + 3$ terms used in the computation. Denote the size of the leading order truncated term by E_{J+3}. While it may be possible to obtain E_{J+3} for any time

varying system it is likely to be unduly restrictive if based on the most general form of the equations above. Rather consider the $L \times n$ order linear time invariant (LTI) equation

$$\frac{d^n x_l}{dt^n} = \lambda_l x_l, \quad l = 1,...,L$$

where λ_l , $l = 1,...,L$ are the coefficients of a diagonalized system. It has been found empirically by the author that the largest value of the highest order term in the series truncation error, using $J+3$ terms of the series, will have for bound

$$E_{J+3} \le \frac{h^{J+2}(\lambda_{\max})^{\alpha}}{(J+2)!} \quad for\ \alpha = \{1 + \mathrm{int}(\frac{J+1}{n})\} \tag{7}$$

where $\lambda_{\max} = \max[|\lambda_1|,...,|\lambda_L|]$. In the examples that follow, the largest magnitude entry of the matrix *A(t)* at each time step will be used as an estimate of $\lambda_{\max}$. By doing this, the assumption is that this will result in a sufficiently accurate estimate of the truncation error. Equation (7) does not of course represent the total error in the scheme, which would also include the error in the approximation in equation (4).

The solution of the last term in equation (2) is facilitated by the fact that values of the STM at successive time steps are already available, which can be used to numerically evaluate the integral term. For constant step size *h*, write

$$\begin{aligned}
\xi_m^0 &= \Phi(t_m + h, t_m) B(t_m) u(t_m) \\
\xi_m^{-1} &= \Phi(t_m + h, t_{m-1}) B(t_{m-1}) u(t_{m-1}) \\
&= \Phi(t_m + h, t_m) \xi_{m-1}^0 \\
&\vdots \\
\xi_m^{-I+2} &= \Phi(t_m + h, t_m) \xi_{m-1}^{-I+3}
\end{aligned}$$

A recursion for computing the vectors $\xi_m^1,...,\xi_m^{-I+2}$ can easily be constructed from these relations. The vector of integrals is evaluated by defining $\mathrm{I} \in R^N$ where

$$\mathrm{I} = \int_{t_m}^{t_m+h} \Phi(t_m + h, T) B(T) u(T) dT$$

or an approximation of this integral

$$\mathrm{I} \approx \int_0^h \sum_{i=1}^{I} \hat{c}_i \varsigma^{i-1} / h^{i-1} d\varsigma \tag{8}$$

where *I-1* is the order, $\varsigma = T - t_m$ and the vector $\hat{c}_i$ has the functional dependence $\hat{c}_i = \hat{c}_i(\xi_m^1,...,\xi_m^{-I+2})$, $\xi_m^1,\ldots,\xi_m^{-I+2} \in R^N$, $\hat{c}_i \in R^N$. To illustrate the procedure, the approximation (equation (8)) can be evaluated for orders zero and one by

O(0) scheme: $\quad \mathrm{I} \approx h\hat{c}_1 \quad$ where $\hat{c}_1 = \xi_m^0$

O(I) scheme: $\quad \mathrm{I} \approx h(\hat{c}_1 + \tfrac{1}{2}\hat{c}_2) \quad$ where $\hat{c}_1 = \xi_m^0$, $\hat{c}_2 = \xi_m^1 - \xi_m^0$

This completes development of the method for an LTV system. The next section outlines modification of the procedure for an NLTV system.

Section 2.3. Approximation of the STM for an NLTV system. Consider now the case in which

$$\dot{x} = \tilde{A}(t,x)x + B(t)u(t) \tag{9}$$

Suppose that the linear and nonlinear parts of the matrix $\tilde{A}(t,x)$ can be expressed as

$$\tilde{A}(t,x) = A(t) + F(x)$$

where

$$\tilde{a}_{n,l}^{-1} = a_{n,l}(t_m - h) + f_{n,l}^{-1} \quad , \quad \tilde{a}_{n,l}^{0} = a_{n,l}(t_m) + f_{n,l}^{0} \quad ,\cdots, \quad \tilde{a}_{n,l}^{r} = a_{n,l}(t_m + rh) + f_{n,l}^{r}$$

where

$$f_{n,l}^{-1} = f_{n,l}(x(t_m - h)) \quad , \quad f_{n,l}^{0} = f_{n,l}(x(t_m)) \quad ,\cdots, \quad f_{n,l}^{r} = f_{n,l}(x(t_m + rh))$$

and the coefficients $a_{n,l}^1, a_{n,l}^0,\ldots$ are the time varying part. A local approximation of $\tilde{A}(t,x)$ can be found using equation (4) where now $c_{i\,n,l} = c_{i\,n,l}(\tilde{a}_{n,l}^0,\ldots,\tilde{a}_{n,l}^{-I})$. Because the construction of these coefficients is a crucial step additional explanation is required. The nonlinear terms $\tilde{a}_{n,l}$ are again approximated by polynomial expressions. For example, orders zero and one are given by

O(0) scheme: $\quad \tilde{a}_{n,l}(\tau) = c_{1\,n,l} \quad$ where $c_{1\,n,l} = a_{n,l}^0 + f_{n,l}^0$

O(I) scheme: $\quad \tilde{a}_{n,l}(\tau) = c_{1\,n,l} + c_{2\,n,l}\tau / h$

where

$$c_{1\,n,l} = a_{n,l}^0 + f_{n,l}^0 \quad , \quad c_{2\,n,l} = a_{n,l}^1 - a_{n,l}^0 + f_{n,l}^1 - f_{n,l}^0$$

and $\tau \in (0,h)$. The value of the solution $x(t_m + h)$ has not yet been computed and therefore the function $f_{n,l}^1$ is not yet known. This term is computed by extrapolation, that is $f_{n,l}^1 = f_{n,l}(\bar{x}^1)$, where $\bar{x}^1 = g(x^0,\ldots,x^{-I+1})$ is derived from previous solutions. The rest of the method follows that outlined in the previous section.

Section 3.0. Examples. In this section solutions of LTV and NLTV systems are computed.

Example 3.1. This example computes the following second order system

$$\ddot{x} = t^4 x\,, \quad x(0) = 0,\ \dot{x}(0) = 1$$

or

$$\dot{\xi} = A\xi$$

where

$$A = \begin{bmatrix} 0 & 1 \\ t^4 & 0 \end{bmatrix}$$

computed on $t : 0 \to 1$. The exact solution is given by the series

$$x(t) = \sum_{k=1}^{\infty}\left(\prod_{m=1}^{k}\frac{1}{(6m-5)(6m-6)}\right)t^{6k-5}$$

$$\dot{x}(t) = \sum_{k=1}^{\infty}\left(\prod_{m=1}^{k}\frac{1}{(6m-5)(6m-6)}\right)(6k-5)\,t^{6k-6}$$

Figure 1 presents the error in the O(I), O(II), O(III) and O(IV) schemes compared with the second order implicit Euler method. Convergence to the exact solution is shown as the order of the approximation of *A(t)* increases. The O(IV) scheme is exact to machine accuracy since the time variation of the matrix *A(t)* is exactly captured with a 4th order polynomial.

This example illustrates one of the advantages of this method. If the time variation is sufficiently modeled with the order polynomial used, the method is nearly insensitive to the size of the time step. Using the O(IV) scheme in this example, any time step size is permissible since the exact solution is being computed. This aspect will be explored more fully in the next example.

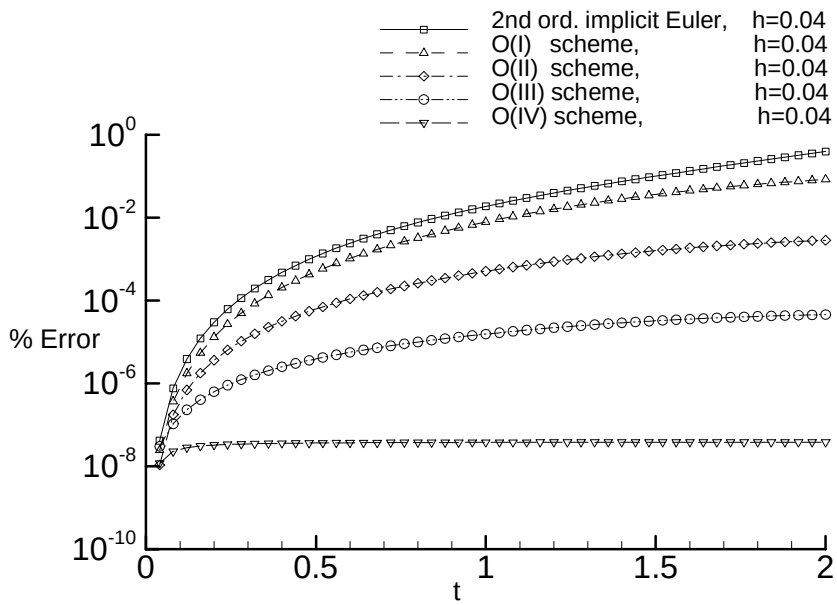

Figure 1. Percent error.

Example 3.2. This example computes the response of a pitch/plunge apparatus initially oscillating as a sinusoid, due to a change in the torsion stiffness to ¼ its original value. The plunge and initial torsion stiffness are $K_h = 1.21\times10^6\, N/m$ and $K_{\alpha 1} = 6.68\times10^3$ $N-m/rad$. The mass is $m_h = 26.64\, kg$, the mass pitch moment of inertia is $I_{c/4} = 0.086\, kg-m^2$, and the static offset is $s_\alpha = 0.378\, kg-m$. These values are for an actual system set up to study transonic flutter.[3] The functional variation of the torsion stiffness is given by

$$K_\alpha = K_{\alpha 1}(1-f) + K_{\alpha 2} f \qquad t_1 \le t \le t_1 + \Delta t$$

where

$$f = \tfrac{1}{2}(1-\cos[\pi(t-t_1)/\Delta t])$$

and $t_1 = 0.24\,\text{sec}$, $\Delta t = 0.01\,\text{sec}$. The natural frequencies of the plunge and torsion modes are *205.4 rad/sec* and *299.3 rad/sec* for $t \le t_1$. The second order system is diagonalized, resulting in the equation

$$\ddot{q} = \hat{A}q + \hat{B}u\,.$$

The matrix $\hat{A}(t)$ has the values

$$\hat{A} = \begin{bmatrix} -42189 & 0 \\ 0 & -89580 \end{bmatrix} \quad for\ t \le t_1, \qquad \hat{A} = \begin{bmatrix} -39298 & 10533 \\ 10533 & -50876 \end{bmatrix} \quad for\ t \ge t_1 + \Delta t$$

and

$$\hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}.$$

The generalized variables q_1 and q_2 roughly correspond to pitch and plunge. These equations are written in state variable form and solved using equations (5) and (6) with a sufficient number of terms adjusted at each time step to ensure $E_{J+3} \le 1\times10^{-7}$ (see equation (7)).

Computations are performed with h = 0.0005, 0.001, 0.002, and 0.004 *sec* (approximately 64 to 8 time steps/pitch cycle). The ensuing results are compared with an integration using a variable time step size 4th order Runge-Kutta (R-K) scheme with tolerance of 1×10^{-7}. The resulting time traces are shown in Figures 2-11. Figures 2-7 are computed with zero input, i.e. $u_1 = u_2 = 0$, but with initial conditions $q_1(0) = q_2(0) = 1$. Figures 8-11 are computed with initial conditions $q_1(0) = 0,\ \ q_2(0) = 0$, but with an impulse input for u_2 centered at $t = .1\,\text{sec}$.

Figure 2 gives an overview of the time history of variable q_2 solved with the R-K method, showing the torsion stiffness change at time $t_1 = 0.24\,\text{sec}$ and continuing afterwards for about 40 cycles. The R-K solution required an average time step size of $h_{ave} = 0.0001\,\text{sec}$ to achieve the desired error tolerance and about 10000 time steps over the interval shown.

Figure 3 compares the O(III) scheme solution at $h = 0.004\,\text{sec}$ with the R-K simulation. Even after nearly 50 cycles of oscillation there is no discernable difference in the solutions. A solution computed with the 2nd order implicit Euler method using $h = 0.002\,\text{sec}$ is compared with the 4th order variable time step R-K solution in Figure 4. Although the very large phase error of the Euler method for this problem makes that solution unusable, it does illustrate the difficulty in obtaining accurate solutions over a very large time span. By contrast, all of the solutions shown using the present scheme are quite accurate.

The solutions in Figures 5-7 using the O(I) to the O(III) scheme are computed at successively smaller time step sizes. Figure 5 presents solutions using the O(I) scheme at successive time step sizes. Figure 6 presents solutions using the O(II) scheme, while Figure 7 shows solutions using the O(III) scheme, each at successively smaller time step sizes. Each set of solutions demonstrates the convergence of the present schemes to the 4th order R-K solution as the time step size is decreased. The O(I) solution in Figure 5 converges, albeit slowly. As expected, the solution using the O(III) scheme seen in

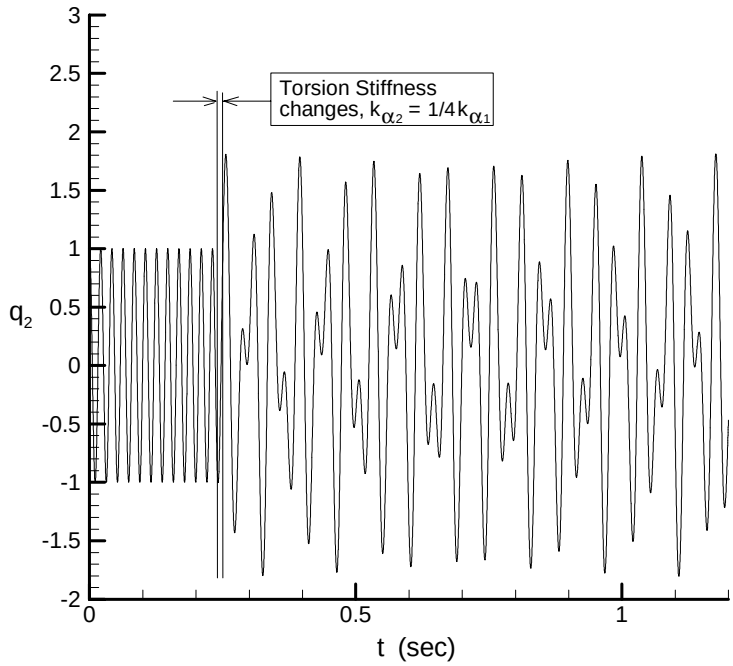

Figure 2. Second mode response to smooth change in torsion stiffness.

Figure 7, converges much more rapidly with time step reduction to the R-K solution than the solution using the O(I) scheme seen in Figure 5.

The last simulations using the pitch/plunge set up, seen in Figures 8-11, show the response due to an impulse input. Figure 8 shows the overall solution and the impulse input that initiates the dynamic response. The discrete impulse input centered at $t = 0.1\,\text{sec}$ is shown in Figure 9 at successive time step sizes. That figure illustrates the coarseness of the approximation of the impulse at the largest time step. Even at the largest time step size, at which the impulse is defined by only three time steps, the relative insensitivity of the O(III) scheme to time step size can be seen in the results of Figures 10-11.

The last three examples apply the method to NLTV systems.

Example 3.3. The equation

$$\dot{x}_1 = x_2\ , \quad \dot{x}_2 = -cx_1 + g\cos bt - dx_2 - ax_1^n \tag{10}$$

is solved. When $n = 2$ this equation is the Helmholtz oscillator arising in the modeling of ship capsizing.[4] When $n = 3$ the Duffing oscillator can be modeled, arising in the study of electronic oscillators[5], or structural dynamics in which there is nonlinear stiffness.[6] The parameters are $c = -1,\ a = b = 1,\ g = 0.3,\ d = 0.15,\ n = 3$.

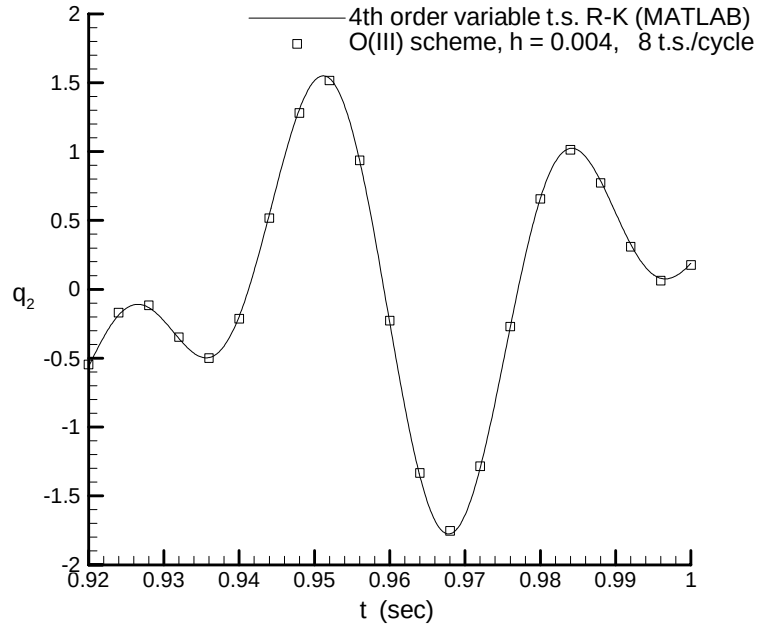

Figure 3. Second mode response to smooth change in torsion stiffness.

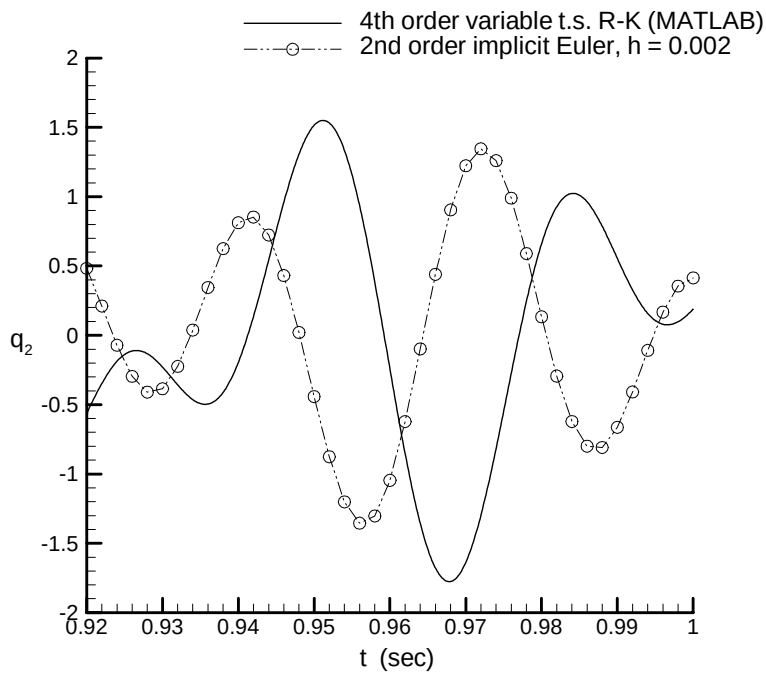

Figure 4. Second mode response to smooth change in torsion stiffness.

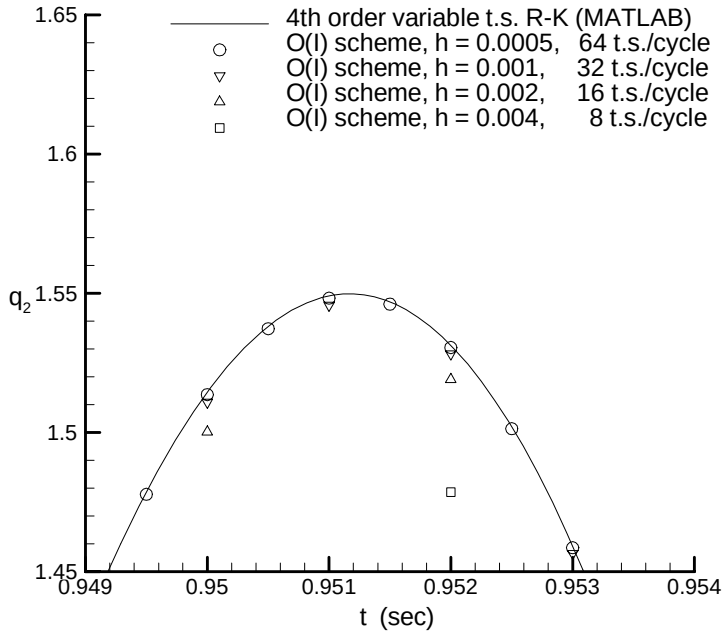

Figure 5. Second mode response to smooth change in torsion stiffness.

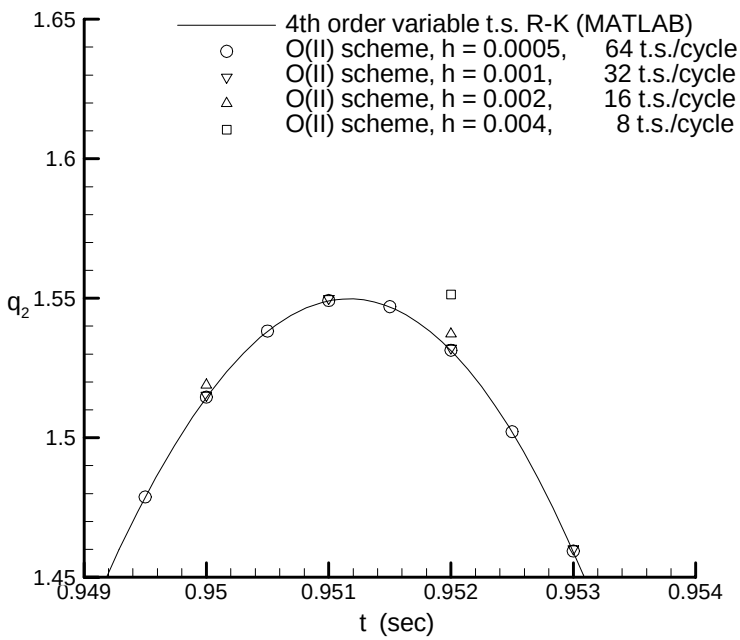

Figure 6. Second mode response to smooth change in torsion stiffness.

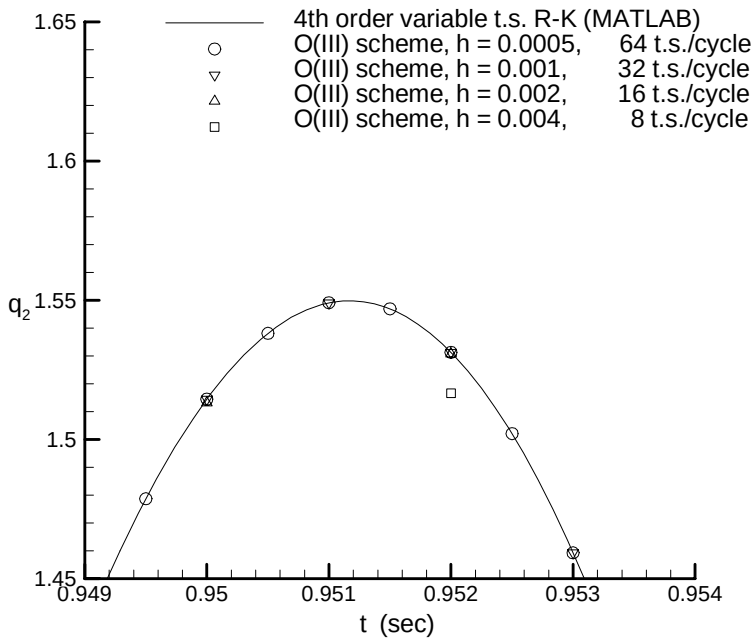

Figure 7. Second mode response to smooth change in torsion stiffness.

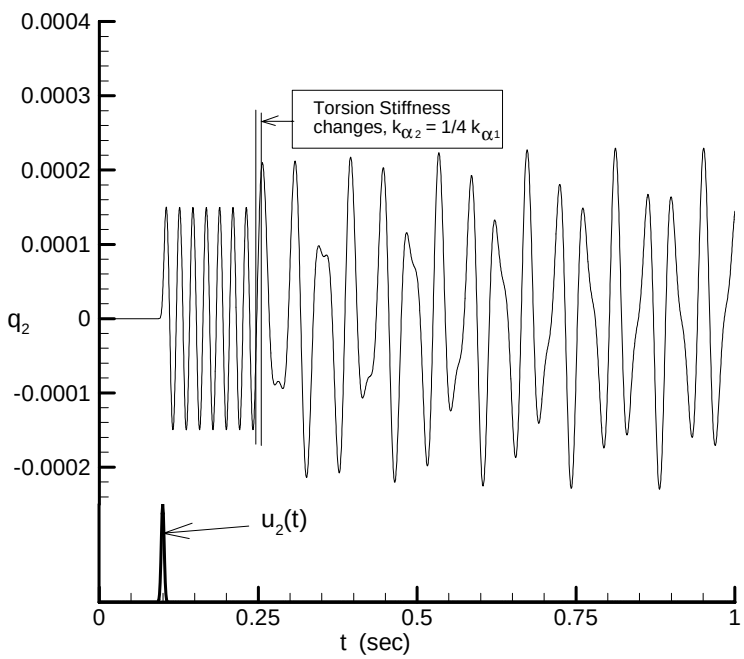

Figure 8. Impulsive input and second mode response to input $u_2(t)$.

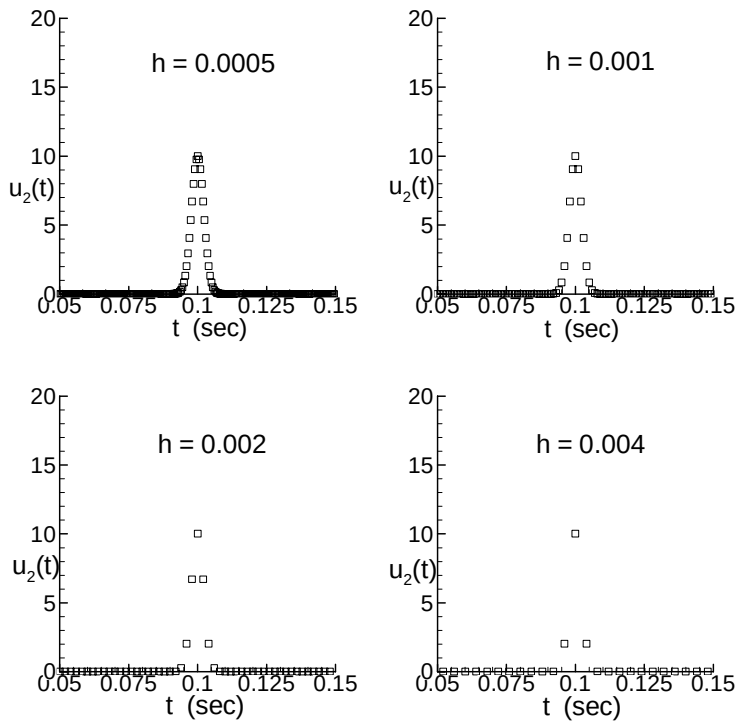

Figure 9. Discrete input $u_2(t)$.

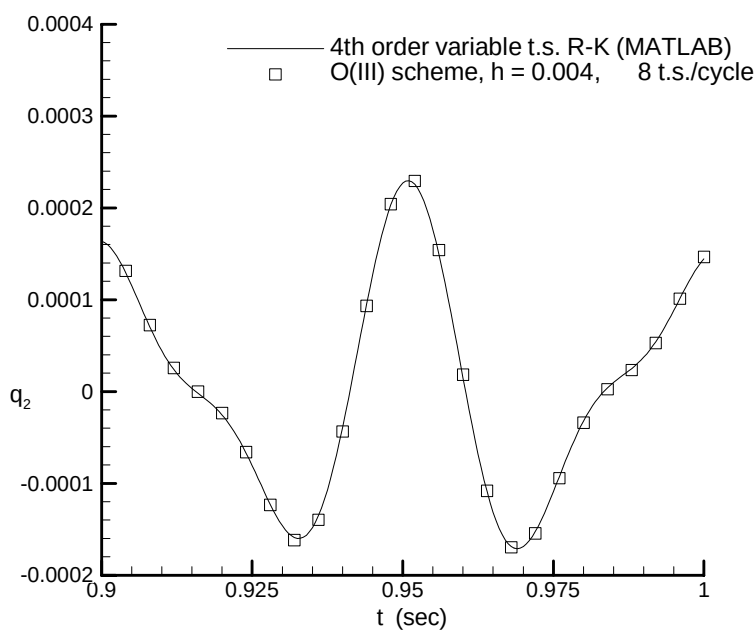

Figure 10. Second mode response to input $u_2(t)$.

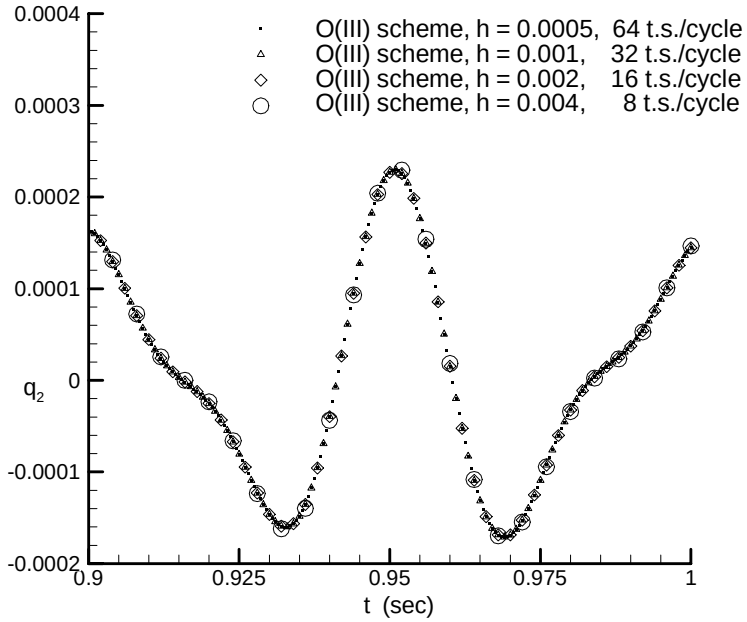

Figure 11. Second mode response to input $u_2(t)$.

Solutions of this equation using the O(0) – O(IV) schemes with initial condition $x_1(0) = -1,\ x_2(0) = 1$ are in Figure 12 a)-e). The tolerance in the present schemes is set at $E_{J+3} \le 1\times10^{-7}$. A solution is also computed with the variable order multi-step solver ode15s in MATLAB, with tolerance of 1×10^{-7}. The variable order solution takes about 4000 time steps over the interval shown, with an average time step size of 0.012. As the order of the present scheme is increased, the solution converges to the reference solution: as shown in Figure 12 e) the O(IV) scheme with h = 0.05 is identical to the reference solution. However, as shown in Figure 13, with just a slight change in the initial condition to $x_1(0) = -1$, $x_2(0) = 1.001$ the O(IV) scheme with h = 0.05 shows divergence from the reference solution toward the end of the simulation. Time step reduction to h = 0.01 is required using the O(IV) scheme for complete convergence.

Example 3.4. Equation (10) is solved with values now given by $c = -\frac{1}{2}$, $a = \frac{1}{2}$, $b = 0.79$, $g = 0.095$, $d = 0.1$, $n = 3$. This is the twin well oscillator with chaotic motion between attractors.[5] For the reference solution, the MATLAB ode15s solver is employed. Since this case is much more sensitive to numerical error than the last, the tolerance is now set at 1×10^{-14}. At this level of tolerance, the average time step size is $h_{ave} = 0.0005$ with 600,000 time steps required. This solution and that computed using the O(V) scheme with h = 0.002, and $E_{J+3} \le 1\times10^{-12}$ are shown in Figure 14. The O(V) scheme and the MATLAB solution diverge only slightly and only toward the end of the simulation.

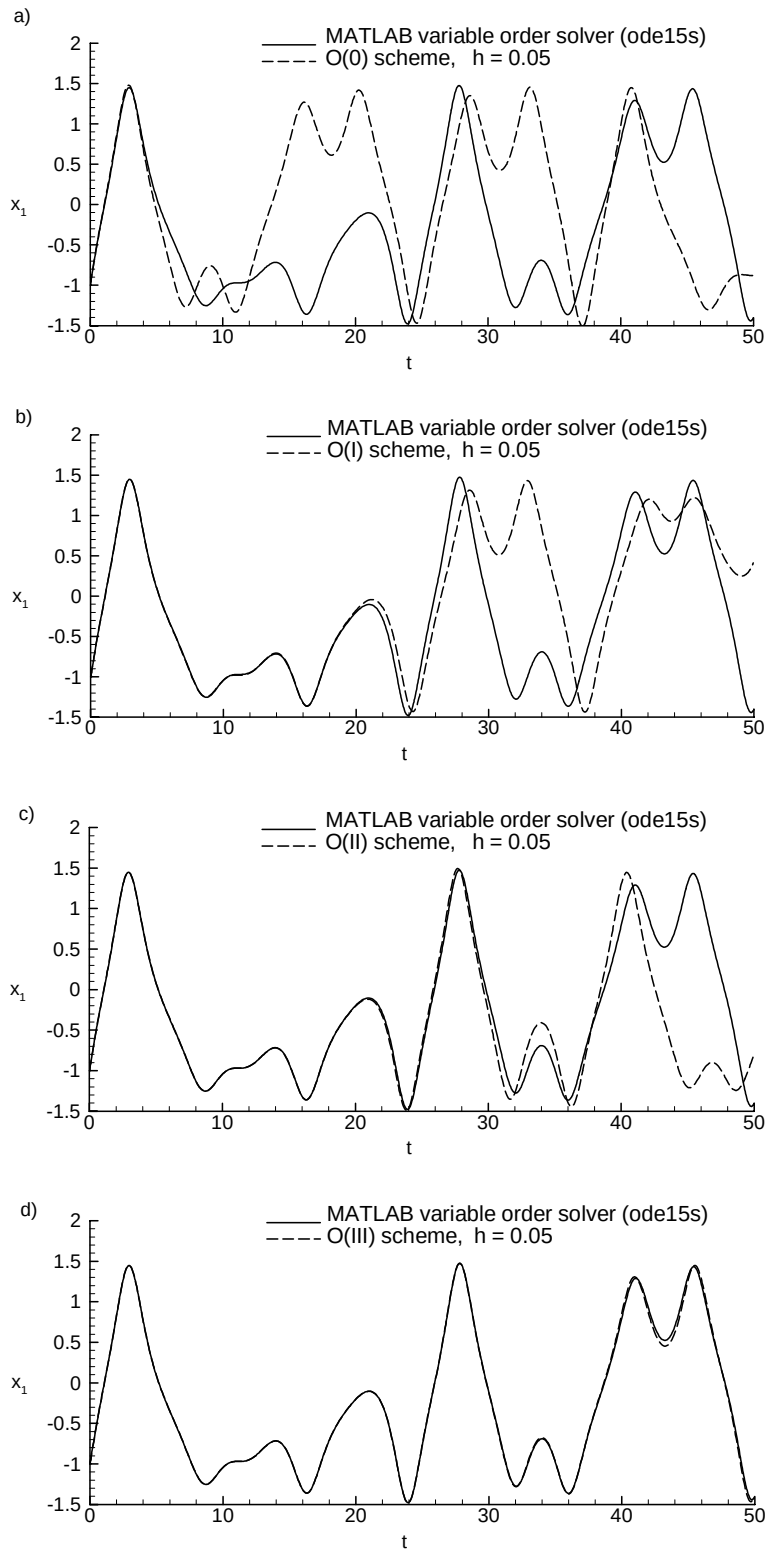

Figure 12 a)-e). Numerical solution of the Duffing equation, x(0)=[-1,1.].

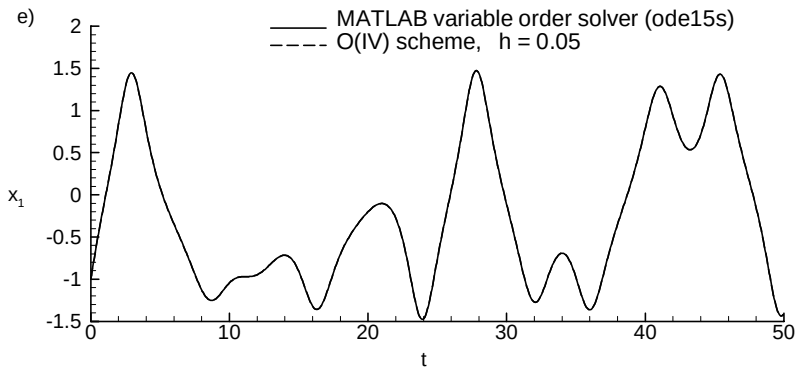

Figure 12. Concluded.

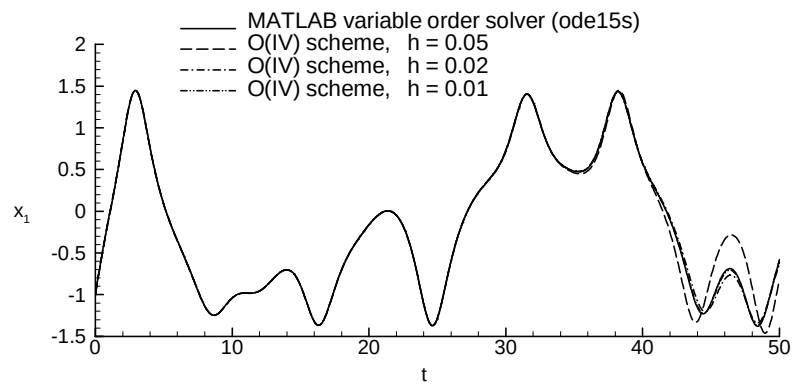

Figure 13. Numerical solution of the Duffing equation, x(0)=[-1,1.001].

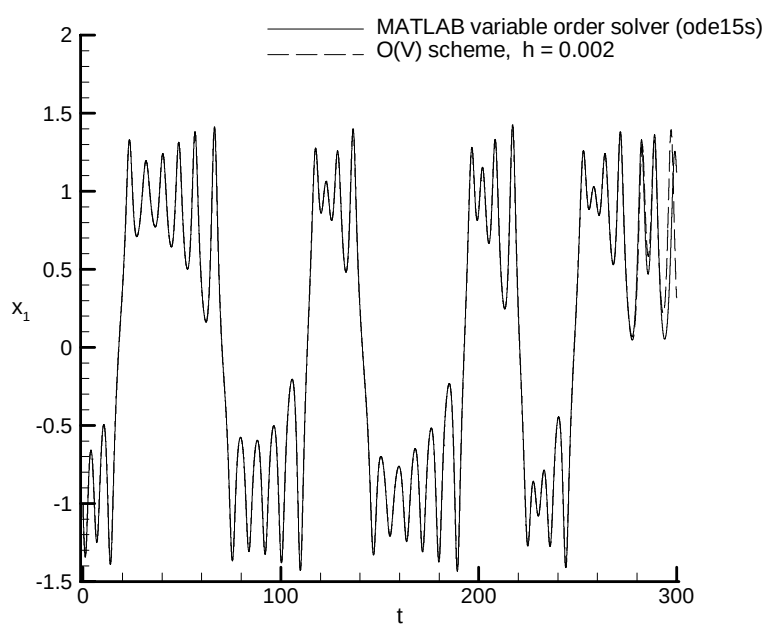

Figure 14. Numerical solution of the twin well equation, $x(0) = [-1,-0.5]$.

Example 3.5. The van der Pol equation

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x$$

is computed with $\mu = 10$. At this value of μ, the equation is only moderately stiff. Figure 15 shows the solution in the region t = 6-11 in which a transition in the solution occurs; before t = 6 the solutions are nearly constant and are identical. The result obtained from the O(III) scheme, with h = 0.05 and $E_{J+3} \le 1\times10^{-6}$, is compared with that from a variable order multi-step solver (ode15s in MATLAB) with tolerance set at 1×10^{-6}. Slight differences in the solutions appear only in the area in which there are large solution gradients. The nearly identical results indicate that, at the large time step used, the present O(III) scheme computes the rapidly changing solution well.

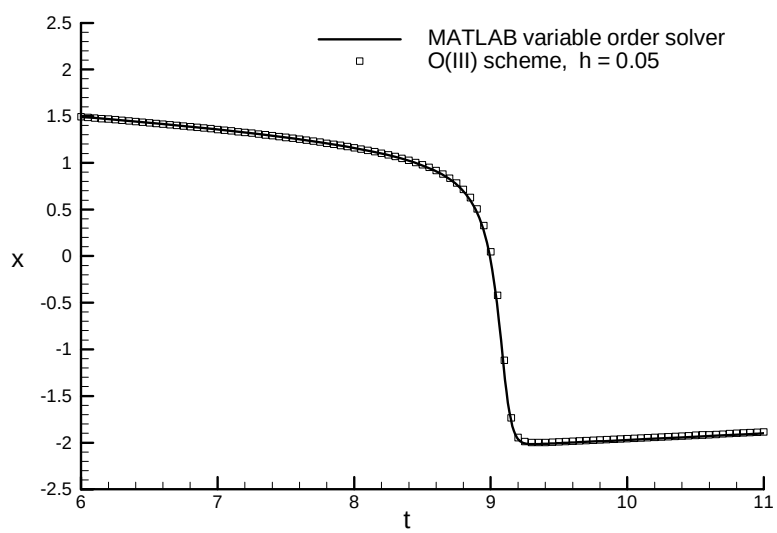

Figure 15. Numerical solution of the van der Pol equation, μ = 10.

Section 4.0. Conclusions. A variable order method of integrating initial value ordinary differential equations that is based on the state transition matrix (STM) has been developed. Although the STM solution formally is valid only for linear equations, the discrete numerical method can be applied to both linear time variant and to nonlinear systems of equations. When the time variation of a linear system can be exactly modeled by a polynomial expansion, the scheme will give an exact result for large time step size. Accurate solutions in all the cases shown can be achieved for linear time variant systems at large time step sizes. The type of nonlinear system is limited to those that can be modeled within a state space frame work, that is, in which the nonlinearity is contained in the *A(t,x), B(t,x)* matrix terms. Several nonlinear problems such as the Duffing equation and twin well attractor, exhibiting chaotic behavior, have also been computed. Higher

order solutions of these equations compare well with reference solutions performed with the standard 4th order Runge-Kutta and variable order methods.

The present method has been developed for integration of modal structural, flight dynamics and control system equations in conjunction with a computational fluid dynamics solver, an application for which it is ideally suited. Applications might include the simulation of structural dynamics with time varying parameters such as stiffness, mass or geometry, simulation of a fully adaptive control system, or flight dynamics with variation in flight parameters. Other applications are also being considered. Apart from its use in a numerical integration scheme, it offers a generalized numerical technique for the computation of the STM for any time varying linear system, among other possible applications.

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13. ABSTRACT *(Maximum 200 words)*
A variable order method of integrating initial value ordinary differential equations that is based on the state transition matrix has been developed. The method has been evaluated for linear time variant and nonlinear systems of equations. While it is more complex than most other methods, it produces exact solutions at arbitrary time step size when the time variation of the system can be modeled exactly by a polynomial. Solutions to several nonlinear problems exhibiting chaotic behavior have been computed. Accuracy of the method has been demonstrated by comparison with an exact solution and with solutions obtained by established methods.

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