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KEEPING A SPACECRAFT ON THE SUN-EARTH LINE

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Measurements of Earth's atmosphere as it occults sunlight can be obtained advantageously from a spacecraft placed in the proximity of the Sun-Earth Lagrange point $\mathcal{L}_2$. Maintaining the condition of continuous solar occultation by all parts of the atmospheric disk requires that the displacement of the spacecraft perpendicular to the Sun-Earth line remains less than 200 km. However, the gravitational force exerted by the Earth's moon must be negated by propulsion in order to meet this rather tight constraint. We provide an estimate of propulsive force needed to keep the spacecraft coincident with $\mathcal{L}_2$, and a second estimate in which the spacecraft is allowed to move along the Sun-Earth line.

INTRODUCTION

The benefits of placing spacecraft in halo orbits about Lagrange or libration points are widely discussed in the literature, as are methods for controlling such orbits. Use of the collinear Sun-Earth Lagrange points in particular for the conduct of Earth and space science missions is reviewed, for example, in Refs. [1], [2], and [3]. Halo orbits, and the more general Lissajous orbits, typically involve displacements from the Sun-Earth line ranging from thousands to hundreds of thousands of kilometers. Such relatively large excursions can not be tolerated for certain missions proposed recently. For instance, as discussed in Ref. [3] it is advantageous to study Earth's atmosphere from the transEarth equilibrium point $\mathcal{L}_2$ via solar occultation in the near infrared spectra. From this unique vantage

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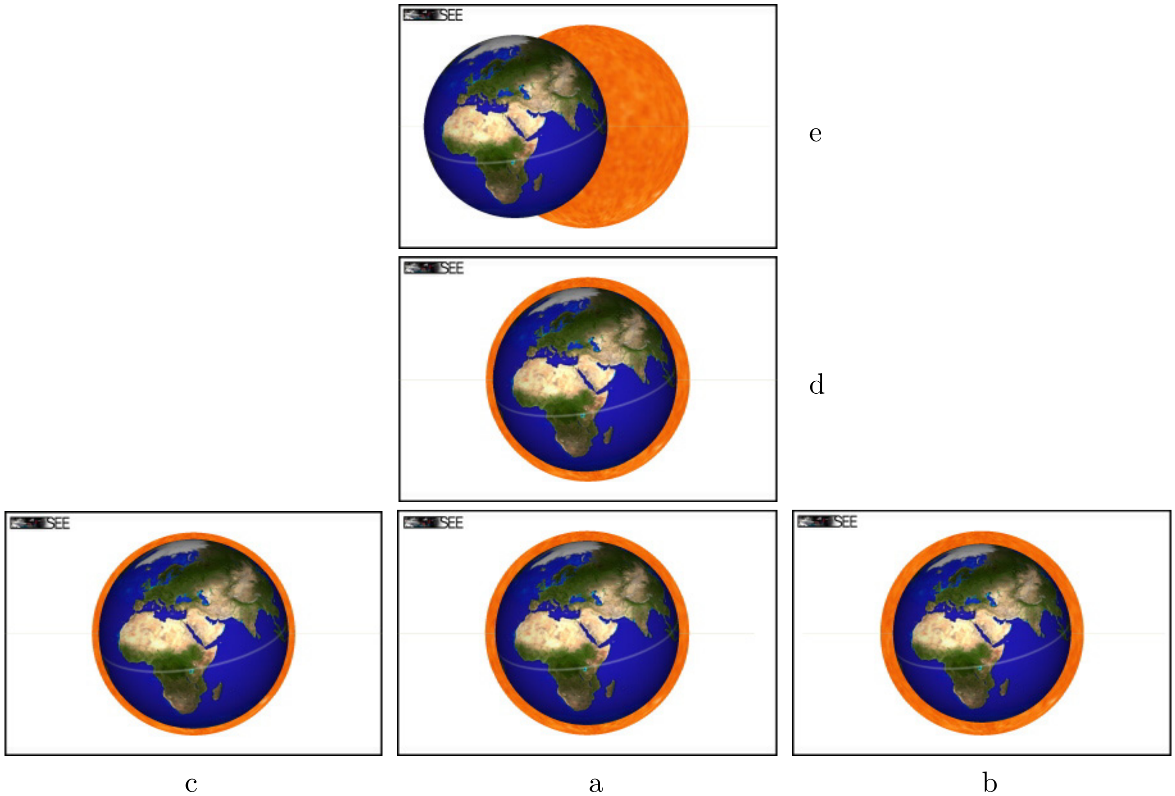

Figure 1: Views from the neighborhood of $\mathcal{L}_2$

point the entire limb of the Earth provides permanent occultation, making it possible to obtain hourly measurements at all latitudes of the atmosphere as the Earth rotates, and produce nearly global high-resolution three-dimensional maps of the geographic distribution of major atmospheric constituent gas species such as CO_2, O_3, O_2, CH_4, H_2O, and N_2O. Such measurements can not be obtained by spacecraft in low Earth orbit, and have never before been achieved.

Maintaining the condition of continuous solar occultation by all parts of the atmospheric disk requires that the displacement of the spacecraft perpendicular to the Sun-Earth line remains less than 200 km. Views of Earth occulting the Sun from various positions in the neighborhood of $\mathcal{L}_2$ are contained in Fig. 1. The rectangular border around each image marks a $1^\circ \times 0.65^\circ$ field of view. Figures 1a, b, and c show Earth centered in front of the solar disk when viewed from three points on the Sun-Earth line; the distance is 1.5082×10^6 km from Earth (the approximate position of $\mathcal{L}_2$) in Fig. 1a, 50,000 km farther away in Fig. 1b, and 50,000 km closer in Fig. 1c. In Figs. 1d and e the vantage point is 1.5082×10^6 km from Earth along the Sun-Earth line, and displaced by some distance in a direction perpendicular to the Sun-Earth line, in the ecliptic plane. From a perpendicular displacement of 200 km the Earth is somewhat off center but still completely within the solar disk as displayed in Fig. 1d; however, a majority of Earth's limb fails to occult the Sun from a perpendicular displacement of 5,000 km as one can see in Fig. 1e.

As is well known from the study of the circular restricted three-body problem, a particle at $\mathcal{L}_2$ is considered to be in dynamic equilibrium based on the assumptions that the Sun and Earth orbit their common barycenter circularly and, more importantly, that only these two bodies exert gravitational force on the particle. The gravitational attraction of the Earth's moon (a fourth body) displaces the point of equilibrium by nearly 5,000 km in a roughly circular path with the period of the lunar cycle; although a spacecraft excursion of this amount in a direction parallel to the Sun-Earth line would not violate the science requirement, displacement of this magnitude in the perpendicular direction is unacceptable. Solutions of the circular restricted three-body problem reveal the existence of nearly rectilinear orbits in the halo family as described in Refs. [4] and [5]; however, the resulting displacement is perpendicular to the line between the primaries, and the perturbation exerted by a fourth body is not accounted for in this result. Consequently, a spacecraft such as the one proposed in Ref. [3] must possess a propulsion system capable of offsetting the lunar gravitational attraction; preliminary design of the system requires an estimate of the associated force per unit mass and corresponding velocity increment.

The requisite estimate is obtained analytically from equations governing the motion of a system of four particles representing the Earth, a spacecraft, the Sun, and the moon. The four bodies are subject to mutual gravitational attraction, and an additional propulsive force is permitted to act on the spacecraft. Upon imposing a kinematical constraint on the spacecraft such that its position must remain coincident with $\mathcal{L}_2$, the propulsive force per unit mass needed to negate the moon's gravity is calculated. An approximate expression for this force is then obtained via expansion in a binomial series, and the force is projected into directions parallel and perpendicular to the Sun-Earth line after neglecting the inclination of the lunar orbit plane relative to the ecliptic. Subsequently, the relationships for the projections are integrated analytically in order to estimate the velocity increment ΔV required of a propulsion system for one lunar cycle.

Requiring the spacecraft to remain coincident with $\mathcal{L}_2$ is admittedly overly restrictive; as mentioned previously, solar occultation by the atmosphere will occur even when the spacecraft is permitted to move a reasonable distance along the Sun-Earth line. Thus, less propellant is needed if thrusters are used solely to offset the lunar perturbing force perpendicular to the Sun-Earth line.

The lunar perturbing force exerted during one lunar cycle is evaluated numerically using positions of the Earth, Sun, and moon specified by published ephemerides, and numerical evaluation of the approximate expressions for the projections is seen to agree well with the exact results. In addition, numerical values of the monthly velocity increments are provided. Simulations are performed to demonstrate the benefits of applying the propulsive force as calculated. Using the published ephemerides to determine the positions of the three celestial bodies, numerical integration of equations of motion is performed without and with the propulsive force to show, in the first case, the large excursion in spacecraft position from $\mathcal{L}_2$ caused by the moon, and, in the second case, a significant reduction in that excursion.

MOTION OF A FOUR-BODY SYSTEM

Figure 2 depicts a system of four particles P_i, each of mass m_i ($i = 1, 2, 3, 4$), moving in a Newtonian reference frame N under the influence of mutual gravitational attraction. We are particularly interested in a system where P_1, P_2, P_3, and P_4 represent respectively the

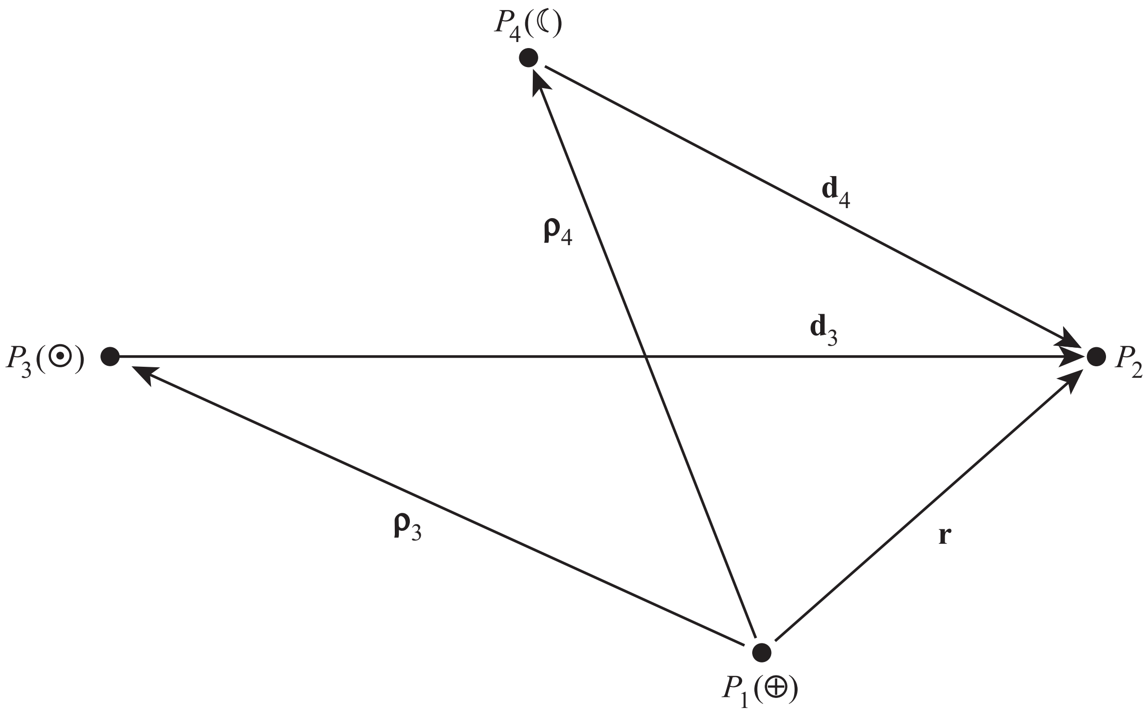

Figure 2: Four-Body System

Earth, a spacecraft, the Sun, and the moon. A propulsive force can be applied to P_2 in order to control its motion.

Using the notation associated with Fig. 8.2 (a) of Ref. [6], $\mathbf{r}$ indicates the position vector from P_1 to P_2, and $\boldsymbol{\rho}_i$ represents the position vector from P_1 to P_i $(i = 3, 4)$. According to Newton's laws of motion and gravitation,

$$^N\mathbf{a}^{P_1} = G\left(\frac{m_2\mathbf{r}}{r^3} + \frac{m_3\boldsymbol{\rho}_3}{{\rho_3}^3} + \frac{m_4\boldsymbol{\rho}_4}{{\rho_4}^3}\right) \tag{1}$$

where $^N\mathbf{a}^{P_1}$ is the acceleration of P_1 in N, r is the magnitude of $\mathbf{r}$, and ρ_i is the magnitude of $\boldsymbol{\rho}_i$ $(i = 3, 4)$. Similarly, $\mathbf{d}_i$ represents the position vector from P_i to P_2 $(i = 3, 4)$, and

$$^N\mathbf{a}^{P_2} = -G\left(\frac{m_1\mathbf{r}}{r^3} + \frac{m_3\mathbf{d}_3}{{d_3}^3} + \frac{m_4\mathbf{d}_4}{{d_4}^3}\right) + \frac{\mathbf{p}}{m_2} \tag{2}$$

where $\mathbf{p}/m_2$ is the propulsive force per unit mass applied to P_2. The motions of P_3 and P_4 are governed by the relationships

$$^N\mathbf{a}^{P_3} = G\left[-\frac{m_1\boldsymbol{\rho}_3}{{\rho_3}^3} + \frac{m_2\mathbf{d}_3}{{d_3}^3} + \frac{m_4\mathbf{r}^{P_3P_4}}{(r^{P_3P_4})^3}\right] \tag{3}$$

and

$$^N\mathbf{a}^{P_4} = G\left[-\frac{m_1\boldsymbol{\rho}_4}{{\rho_4}^3} + \frac{m_2\mathbf{d}_4}{{d_4}^3} - \frac{m_3\mathbf{r}^{P_3P_4}}{(r^{P_3P_4})^3}\right] \tag{4}$$

where $\mathbf{r}^{P_3P_4}$ is the position vector from P_3 to P_4.

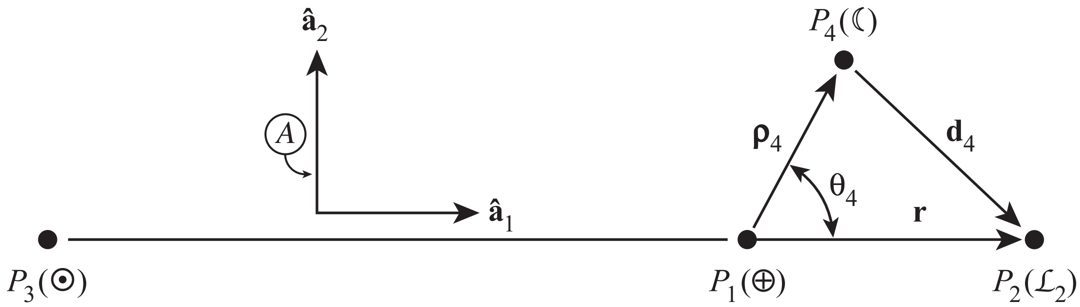

Figure 3: Spacecraft on the Sun-Earth Line

Constrained Motion of P_2

Consider a reference frame A, shown in Fig. 3, in which the line passing through P_3 and P_1 is fixed; P_1 is regarded as fixed in A, but not P_3. Further, a point $\mathcal{L}_2$ is fixed in A, collinear with P_3 and P_1. Frame A moves in frame N with angular velocity ${}^N\boldsymbol{\omega}^A$ and angular acceleration ${}^N\boldsymbol{\alpha}^A$.

After denoting the position vector from P_1 to $\mathcal{L}_2$ as $\mathbf{r}^{P_1\mathcal{L}_2}$, the acceleration ${}^N\mathbf{a}^{\mathcal{L}_2}$ of $\mathcal{L}_2$ in N can be expressed as

$$ {}^N\mathbf{a}^{\mathcal{L}_2} = {}^N\mathbf{a}^{P_1} + {}^N\boldsymbol{\omega}^A \times \left({}^N\boldsymbol{\omega}^A \times \mathbf{r}^{P_1\mathcal{L}_2}\right) + {}^N\boldsymbol{\alpha}^A \times \mathbf{r}^{P_1\mathcal{L}_2} \tag{5} $$

We wish to determine the propulsive force needed to hold P_2 fixed in A and coincident with $\mathcal{L}_2$; that is, somewhere on the Sun-Earth line. In this case ${}^N\mathbf{a}^{P_2} = {}^N\mathbf{a}^{\mathcal{L}_2}$ and $\mathbf{r} = \mathbf{r}^{P_1\mathcal{L}_2}$. In view of Eqs. (5), (2), and (1), we can write

$$ \begin{aligned} {}^N\mathbf{a}^{P_2} - {}^N\mathbf{a}^{P_1} &= {}^N\boldsymbol{\omega}^A \times \left({}^N\boldsymbol{\omega}^A \times \mathbf{r}\right) + {}^N\boldsymbol{\alpha}^A \times \mathbf{r} \\ &= \frac{\mathbf{p}}{m_2} - G\left[\frac{(m_1+m_2)\mathbf{r}}{r^3} + m_3\left(\frac{\mathbf{d}_3}{{d_3}^3} + \frac{\boldsymbol{\rho}_3}{{\rho_3}^3}\right) + m_4\left(\frac{\mathbf{d}_4}{{d_4}^3} + \frac{\boldsymbol{\rho}_4}{{\rho_4}^3}\right)\right] \end{aligned} \tag{6} $$

or

$$ \frac{\mathbf{p}}{m_2} = G\left[\frac{(m_1+m_2)\mathbf{r}}{r^3} + m_3\left(\frac{\mathbf{d}_3}{{d_3}^3} + \frac{\boldsymbol{\rho}_3}{{\rho_3}^3}\right) + m_4\left(\frac{\mathbf{d}_4}{{d_4}^3} + \frac{\boldsymbol{\rho}_4}{{\rho_4}^3}\right)\right] + {}^N\boldsymbol{\omega}^A \times \left({}^N\boldsymbol{\omega}^A \times \mathbf{r}\right) + {}^N\boldsymbol{\alpha}^A \times \mathbf{r} \tag{7} $$

Producing a time history of the propulsive force $\mathbf{p}$ necessary to keep P_2 at rest in A would require apriori assumptions about the motion of the four particles, or solutions of the four second-order differential vector equations (1)–(4). (For example, periodic and quasi-periodic solutions of the coherent restricted four-body problem are obtained in Ref. [7].) From either of these sources one could obtain time histories of $\mathbf{d}_3$, $\mathbf{d}_4$, $\boldsymbol{\rho}_3$, and $\boldsymbol{\rho}_4$, and, with some difficulty, ${}^N\boldsymbol{\omega}^A$ and ${}^N\boldsymbol{\alpha}^A$.

Relative Motion of P_1 and P_2

As an alternative to using Eq. (7) to determine $\mathbf{p}$, one may recast Eq. (6) as

$$ {}^N\mathbf{a}^{P_2} - {}^N\mathbf{a}^{P_1} = \frac{\mathbf{p}}{m_2} - G\left[\frac{(m_1+m_2)\mathbf{r}}{r^3} + m_3\left(\frac{\mathbf{d}_3}{{d_3}^3} + \frac{\boldsymbol{\rho}_3}{{\rho_3}^3}\right) + m_4\left(\frac{\mathbf{d}_4}{{d_4}^3} + \frac{\boldsymbol{\rho}_4}{{\rho_4}^3}\right)\right] \tag{8} $$

Upon realizing that ${}^{N}\mathbf{a}^{P_2} - {}^{N}\mathbf{a}^{P_1}$ is identical to ${}^{N}d^2\mathbf{r}/dt^2$, the second derivative of $\mathbf{r}$ with respect to time in N, we have

$$\frac{{}^{N}d^2}{dt^2}\mathbf{r} + \frac{G(m_1+m_2)\mathbf{r}}{r^3} = \frac{\mathbf{p}}{m_2} - G\left[m_3\left(\frac{\mathbf{d}_3}{{d_3}^3} + \frac{\boldsymbol{\rho}_3}{{\rho_3}^3}\right) + m_4\left(\frac{\mathbf{d}_4}{{d_4}^3} + \frac{\boldsymbol{\rho}_4}{{\rho_4}^3}\right)\right] \tag{9}$$

This can be recognized as Eq. (8.55) of Ref. [6], in the absence of $\mathbf{p}$, when the number of particles n is 4.

Now, suppose we choose

$$\frac{\mathbf{p}}{m_2} = Gm_4\left(\frac{\mathbf{d}_4}{{d_4}^3} + \frac{\boldsymbol{\rho}_4}{{\rho_4}^3}\right) \tag{10}$$

so that Eq. (9) is reduced to

$$\frac{{}^{N}d^2}{dt^2}\mathbf{r} + \frac{G(m_1+m_2)\mathbf{r}}{r^3} = -Gm_3\left(\frac{\mathbf{d}_3}{{d_3}^3} + \frac{\boldsymbol{\rho}_3}{{\rho_3}^3}\right) \tag{11}$$

The result is equivalent to Battin's relationship in the absence of P_4 (the moon) and any propulsive force applied to P_2; in other words, a relationship for relative motion of P_1 and P_2 (Earth and spacecraft), disturbed only by P_3 (the Sun). Because the spacecraft's influence on the motion of the Earth and Sun is negligible, application of propulsive force according to Eq. (10) reduces the problem from that of restricted four-body motion to one of restricted three-body motion, and one may hope to keep P_2 near an unstable equilibrium point $\mathcal{L}_2$ with very little additional propellant. The moon's effect on the Earth and spacecraft, which is considerable, is accounted for by the right hand side of Eq. (10); the moon's effect on the Sun is implicitly reflected in the position vectors $\mathbf{d}_3$ and $\boldsymbol{\rho}_3$ when they are obtained from an accurate ephemeris, but is certainly negligible in view of the size of m_4/m_3.

ESTIMATE OF ΔV

Approximate relationships for the required propulsive force per unit mass expressed in Eq. (10), and for the corresponding velocity increment that must be provided, are obtained as follows. After recognizing that $\boldsymbol{\rho}_4 + \mathbf{d}_4 = \mathbf{r}$, Eq. (10) can be rewritten as

$$\frac{\mathbf{p}}{m_2} = Gm_4\left(\frac{\mathbf{r}-\boldsymbol{\rho}_4}{|\mathbf{r}-\boldsymbol{\rho}_4|^3} + \frac{\boldsymbol{\rho}_4}{{\rho_4}^3}\right) \tag{12}$$

The distance r from Earth to $\mathcal{L}_2$ is approximately 1.5×10^6 km, or four times the distance ρ_4 from Earth to the moon, 384,400 km. Thus, we factor out r from the denominator of the first term

$$\frac{1}{|\mathbf{r}-\boldsymbol{\rho}_4|^3} = [(\mathbf{r}-\boldsymbol{\rho}_4)\cdot(\mathbf{r}-\boldsymbol{\rho}_4)]^{-\frac{3}{2}} = r^{-3}\left[1 - 2\frac{\mathbf{r}\cdot\boldsymbol{\rho}_4}{r^2} + \left(\frac{\rho_4}{r}\right)^2\right]^{-\frac{3}{2}} \tag{13}$$

and define x as

$$x \triangleq -2\frac{\mathbf{r}\cdot\boldsymbol{\rho}_4}{r^2} + \left(\frac{\rho_4}{r}\right)^2 \tag{14}$$

Neglecting the inclination of moon's orbit plane to the ecliptic, about 5°, the largest value of x is found to be about 0.58, whereas the smallest value is approximately -0.45; hence, $-1 < x < 1$, and one can employ the binomial series

$$(1+x)^{-\frac{3}{2}} = 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \cdots \tag{15}$$

to write

$$\left[1 - 2\frac{\mathbf{r} \cdot \boldsymbol{\rho}_4}{r^2} + \left(\frac{\rho_4}{r}\right)^2\right]^{-\frac{3}{2}} = 1 + 3\frac{\mathbf{r} \cdot \boldsymbol{\rho}_4}{r^2} + \cdots \tag{16}$$

where the remaining terms are of second or higher degree in ρ_4/r. Substitution from Eq. (16) into (13) and then into (12) yields

$$\frac{\mathbf{p}}{m_2} \approx Gm_4 \left[\frac{\mathbf{r} - \boldsymbol{\rho}_4}{r^3}\left(1 + 3\frac{\mathbf{r} \cdot \boldsymbol{\rho}_4}{r^2}\right) + \frac{\boldsymbol{\rho}_4}{{\rho_4}^3}\right] \approx Gm_4 \left[\boldsymbol{\rho}_4 \left(\frac{1}{{\rho_4}^3} - \frac{1}{r^3}\right) + \frac{\mathbf{r}}{r^3} + 3\frac{(\mathbf{r} \cdot \boldsymbol{\rho}_4)\mathbf{r}}{r^5}\right] \tag{17}$$

where a term involving $(\rho_4/r)^2$ has been neglected in the final step.

It is convenient to introduce three mutually perpendicular unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ fixed in A, where $\hat{\mathbf{a}}_1$ has the same direction as $\mathbf{r} = \mathbf{r}^{P_1\mathcal{L}_2}$, $\hat{\mathbf{a}}_2$ lies in the ecliptic plane, and $\hat{\mathbf{a}}_3 = \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2$. In view of the negligible inclination of the moon's orbit plane to the ecliptic, $\boldsymbol{\rho}_4$ can be approximated by

$$\boldsymbol{\rho}_4 \approx \rho_4(\cos\theta_4\hat{\mathbf{a}}_1 + \sin\theta_4\hat{\mathbf{a}}_2) \tag{18}$$

where θ_4, the angle between $\mathbf{r}$ and $\boldsymbol{\rho}_4$, goes from 0 to 2π during one lunar synodical period. Because $\mathbf{r} = r\hat{\mathbf{a}}_1$, we can write

$$\frac{\mathbf{p} \cdot \hat{\mathbf{a}}_1}{m_2} = Gm_4 \left[\left(\frac{1}{{\rho_4}^3} + \frac{2}{r^3}\right)\rho_4\cos\theta_4 + \frac{1}{r^2}\right] \tag{19}$$

$$\frac{\mathbf{p} \cdot \hat{\mathbf{a}}_2}{m_2} = Gm_4 \left(\frac{1}{{\rho_4}^3} - \frac{1}{r^3}\right)\rho_4\sin\theta_4 \tag{20}$$

The velocity increment to be supplied by thrusters oriented in the $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ directions can be obtained by integrating Eqs. (19) and (20) with respect to time. The velocity increment in each direction over one lunar orbit is given by

$$\Delta V_i \triangleq \int_0^{2\pi/n_4} \frac{|\mathbf{p} \cdot \hat{\mathbf{a}}_i|}{m_2} dt = 4\int_0^{\pi/2} \frac{|\mathbf{p} \cdot \hat{\mathbf{a}}_i|}{n_4 m_2} d\theta_4 \qquad (i = 1, 2) \tag{21}$$

where n_4 is the mean motion of the moon's orbit, $n_4 \approx \sqrt{Gm_1/{\rho_4}^3}$, and $\theta_4 = n_4 t$. The trigonometric functions in Eqs. (19) and (20) allow setting the upper limit of integration to $\pi/2$ and quadrupling the result. Consequently,

$$\Delta V_1 = 4m_4\sqrt{\frac{G}{m_1\rho_4}}\left[1 + \frac{\pi}{2}\left(\frac{\rho_4}{r}\right)^2 + 2\left(\frac{\rho_4}{r}\right)^3\right] \tag{22}$$

$$\Delta V_2 = 4m_4\sqrt{\frac{G}{m_1\rho_4}}\left[1 - \left(\frac{\rho_4}{r}\right)^3\right] \tag{23}$$

The velocity increment required of the thrusters to offset the lunar perturbation and keep P_2 fixed in A, coincident with $\mathcal{L}_2$, is given by $\Delta V = \Delta V_1 + \Delta V_2$; however, if P_2 is permitted to move parallel to $\hat{\mathbf{a}}_1$, the propulsion system need only supply ΔV_2.

NUMERICAL RESULTS

In what follows, the exact and approximate expressions for propulsive force per unit mass needed to negate the lunar perturbation are evaluated numerically and compared. Numerical values of the corresponding approximate velocity increments are reported for one lunar cycle. Finally, the benefits of applying the propulsive force are demonstrated by numerical integration of equations of motion, without and with the propulsive force applied.

The propulsive force per unit mass $\mathbf{p}/m_2$ required for one month to offset the lunar perturbation is evaluated numerically according to the right hand member of Eq. (10); possible numerical difficulties are avoided by adopting the alternative expression suggested in Eq. (8.61) of Ref. [6]. The position vector $\mathbf{r}$ is given a magnitude of 1.5015 $\times 10^6$ km and the direction of $\hat{\mathbf{a}}_1$. The positions of P_3 (Sun), P_1 (Earth), and P_4 (moon) are obtained from the ephemerides published in Ref. [8] for a 30-day period beginning on the epoch of March 20, 2000, 16h:40m:00s GMT (Greenwich Mean Time), selected so that Earth lies between the Sun and moon, and $\boldsymbol{\rho}_4 \cdot \hat{\mathbf{a}}_2 \approx 0$. The projection $p_1 = (\mathbf{p} \cdot \hat{\mathbf{a}}_1)/m_2$ along the Sun-Earth line is shown on the upper left in Fig. 4 with a solid curve, and is seen to vary between 3.8×10^{-5} m/s^2 and -3.3×10^{-5} m/s^2. The projection $p_2 = (\mathbf{p} \cdot \hat{\mathbf{a}}_2)/m_2$ in the ecliptic and perpendicular to the Sun-Earth line, displayed in the upper right, varies nearly sinusoidally between 2.9×10^{-5} m/s^2 and -3.5×10^{-5} m/s^2. The projection $p_3 = (\mathbf{p} \cdot \hat{\mathbf{a}}_3)/m_2$ perpendicular to the Sun-Earth line and to the ecliptic, contained in the lower left, varies nearly sinusoidally between 2.8×10^{-6} m/s^2 and -3.1×10^{-6} m/s^2, an order of magnitude less than the other two projections. The magnitude $p = \sqrt{\mathbf{p} \cdot \mathbf{p}}/m_2$ is displayed in the plot on the lower right, and is observed to vary between 3.8×10^{-5} and 2.9×10^{-5} m/s^2. The relationships in Eqs. (19) and (20) for the first two projections are evaluated with the numerical values listed in Table 1, and the lunar synodical period is used to convert the independent variable θ_4 to time; the results are depicted with dashed curves in Fig. 4 where it is evident that the expressions furnish reasonable approximations for this particular month.

The velocity increments that must be supplied each month by thrusters aimed in the directions of $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ are simply the areas under the corresponding three curves of Fig. 4; approximate expressions for the first two velocity increments are given by Eqs. (22) and (23). Using the values in Table 1, one obtains $\Delta V_1 = 57$ m/s, and $\Delta V_2 = 49$ m/s. Hence, the monthly velocity increment needed to keep the spacecraft fixed on the Sun-Earth line is estimated to be 106 m/s. However, the science requirements of a telescope for observing Earth's atmosphere are satisfied even if it is permitted to take excursions of several thousand km along the Sun-Earth line. In this case the propulsion system need only supply a monthly velocity increment of 49 m/s, yielding a savings of somewhat more than 50% of the propellant expenditure compared to what must be spent to prevent any lunar-induced excursions.

The benefits of applying a propulsive force in accordance with Eq. (10) are illustrated by numerically integrating, in turn, Eqs. (9) and (11). The positions of the three celestial bodies are once again obtained from the ephemerides in Ref. [8] for a 30-day period, and the initial conditions are the same as those used in connection with Fig. 4.

The initial value of the position vector $\mathbf{r}$ from P_1 to P_2 (the spacecraft) is such that P_2 is coincident with $\mathcal{L}_2$, a point collinear with P_3 and P_1; $\mathcal{L}_2$ is taken to be the second

Table 1: ASTRONOMICAL PARAMETERS

ρ_4	384,400 km
r	1.50151×10^6 km
Gm_4	4.903×10^3 km^3/s^2
Gm_1	3.986×10^5 km^3/s^2

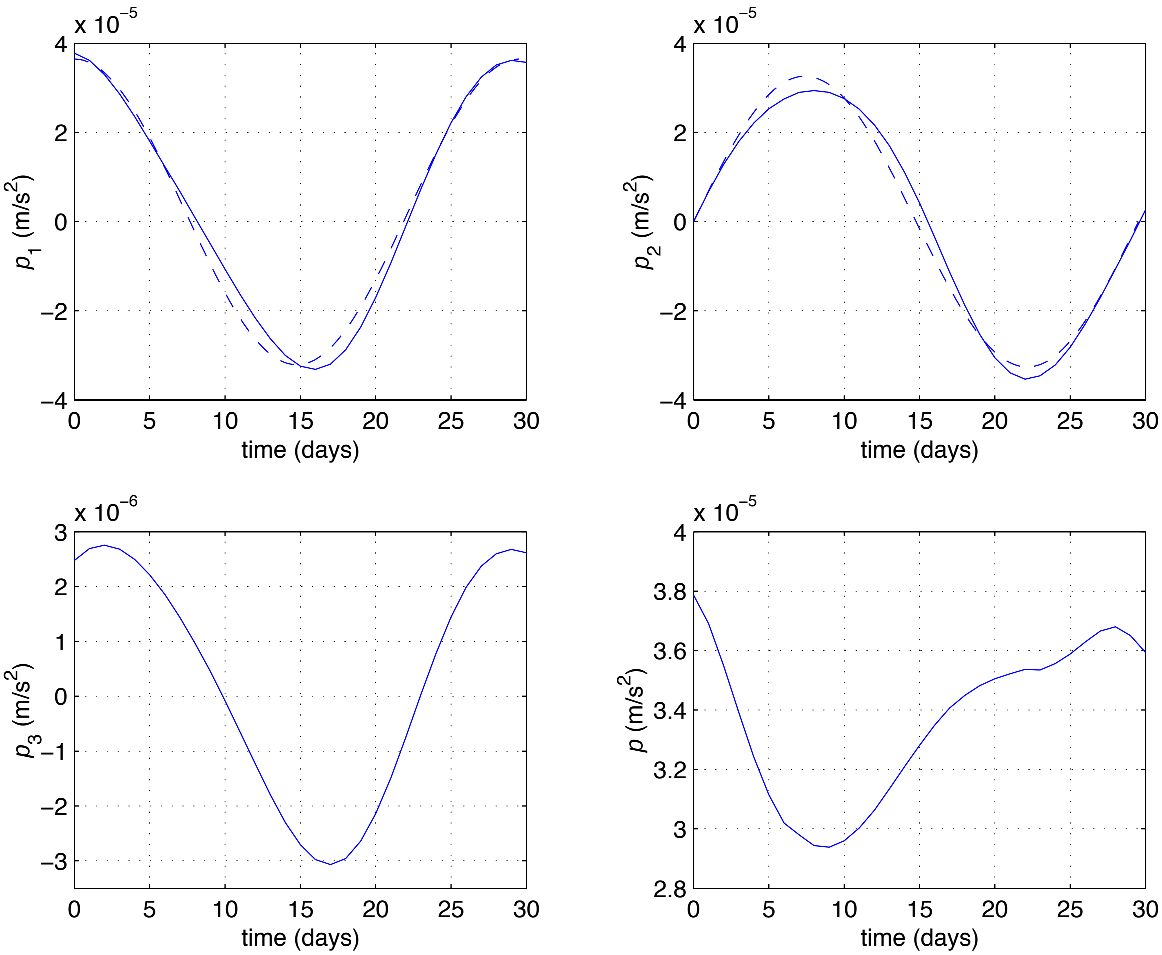

Figure 4: Lunar Perturbing Force Per Unit Mass

Lagrange point based upon the circular restricted three-body model, and the assumption that the mass of P_1 is the sum of the masses of the Earth and moon. As demonstrated in Ref. [9], this yields a more accurate equilibrium position than when the moon's mass is neglected. Also required is an initial value for ${}^N d\mathbf{r}/dt$, given by the general relationship

$$\frac{{}^N d}{dt}\mathbf{r} = {}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1} \tag{24}$$

where ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are the velocities in N of P_1 and P_2, respectively. Because P_1 and $\mathcal{L}_2$ are regarded as fixed in reference frame A, and it is desired that P_2 be coincident with $\mathcal{L}_2$ and fixed in A, the velocities of the two points are related by

$${}^N\mathbf{v}^{P_2} = {}^N\mathbf{v}^{P_1} + {}^N\boldsymbol{\omega}^A \times \mathbf{r}^{P_1\mathcal{L}_2} \tag{25}$$

One may obtain an estimate of ${}^N\boldsymbol{\omega}^A$ by assuming that P_1, P_3, and P_4 all move in the same plane, and therefore ${}^N\boldsymbol{\omega}^A$ is parallel to $\hat{\mathbf{a}}_3$. With the aid of a mechanical analog involving a three-bar linkage and sliding mechanisms, and a moderate amount of kinematical analysis, it can be shown that in the special circumstance when P_3, P_1, and P_4 are collinear,

$${}^N\boldsymbol{\omega}^A(t_0) = \frac{\left(\dot{\nu}_{\oplus} r^{P_3B} \mp \dot{\nu}_m r^{BP_1}\right)}{\rho_3}\hat{\mathbf{a}}_3 \tag{26}$$

where r^{P_3B} is the distance from P_3 to the barycenter B of P_1 and P_4, r^{BP_1} is the distance from B to P_1, $\dot{\nu}_{\oplus}$ is the inertial angular speed of a reference frame in which the line passing through P_3 and B is fixed, and $\dot{\nu}_m$ is the inertial angular speed of a reference frame in which the line passing through P_1, B, and P_4 is fixed. The negative sign applies when P_1 is between P_3 and P_4, whereas the positive sign is used when P_4 is between P_3 and P_1. Hence

$$\frac{{}^N d}{dt}\mathbf{r}(t_0) = {}^N\boldsymbol{\omega}^A(t_0) \times \mathbf{r}^{P_1\mathcal{L}_2} = \left(\dot{\nu}_{\oplus} r^{P_3B} \mp \dot{\nu}_m r^{BP_1}\right)\frac{r^{P_1\mathcal{L}_2}}{\rho_3}\hat{\mathbf{a}}_2 \tag{27}$$

where $r^{P_1\mathcal{L}_2}$ is simply the magnitude of $\mathbf{r}^{P_1\mathcal{L}_2}$. The time rate of change of the true anomaly of the Earth-Moon barycenter, $\dot{\nu}_{\oplus}$, is evaluated with Eq. (2.5-3) of Ref. [10] in order to account for eccentricity and the barycenter's position in its heliocentric orbit. Likewise, eccentricity and orbital position are included in the evaluation of $\dot{\nu}_m$.

Using initial values of $\mathbf{r}$ and ${}^N d\mathbf{r}/dt$ obtained in the manner just described, Eqs. (9) are integrated numerically for a 30-day period with no propulsive force ($\mathbf{p} = \mathbf{0}$). The excursion of P_2 from $\mathcal{L}_2$ is indicated with the position vector $\mathbf{r}^{\mathcal{L}_2P_2} = \mathbf{r} - \mathbf{r}^{P_1\mathcal{L}_2}$, shown in one-day increments in Fig. 5, in three orthogonal views. The plot in the upper left displays $\mathbf{r}^{\mathcal{L}_2P_2} \cdot \hat{\mathbf{a}}_1$ versus $\mathbf{r}^{\mathcal{L}_2P_2} \cdot \hat{\mathbf{a}}_2$. The other two views in Fig. 5 involve $\mathbf{r}^{\mathcal{L}_2P_2} \cdot \hat{\mathbf{a}}_3$. In each view, the location of $\mathcal{L}_2$ is marked by the heavy black circle.

The moon is responsible for a large part of the behavior shown in Fig. 5, a fact demonstrated in a crude fashion by using the same initial values as before to begin a numerical integration of Eqs. (11). The results are shown in Fig. 6; upon comparison to Fig. 5 it can be seen that the excursion in the $\hat{\mathbf{a}}_1$-$\hat{\mathbf{a}}_2$ plane is reduced by a factor of 5, and by a factor of more than 100 in the $\hat{\mathbf{a}}_3$ direction. Thus, application of propulsive force to P_2 according to Eq. (10) goes a long way toward canceling the effect of the moon's gravitation and allowing the spacecraft to remain near a point on the Sun-Earth line. It is clear from Fig. 6 that a feedback system is needed to control the spacecraft position, otherwise natural motion will in a very few days carry it farther than 200 km from the Sun-Earth line.

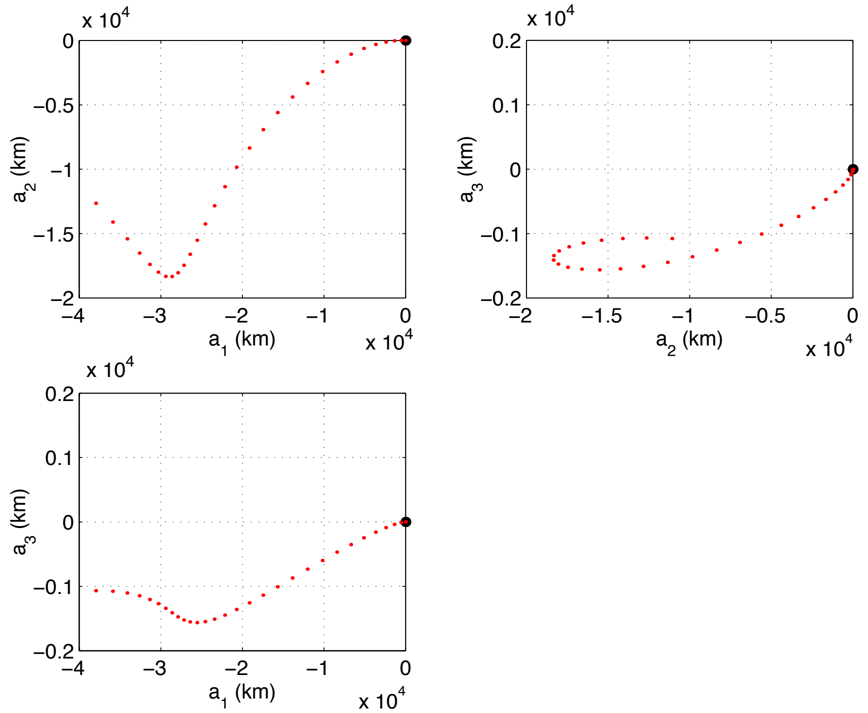

Figure 5: Spacecraft Motion In the Presence of Lunar Perturbation

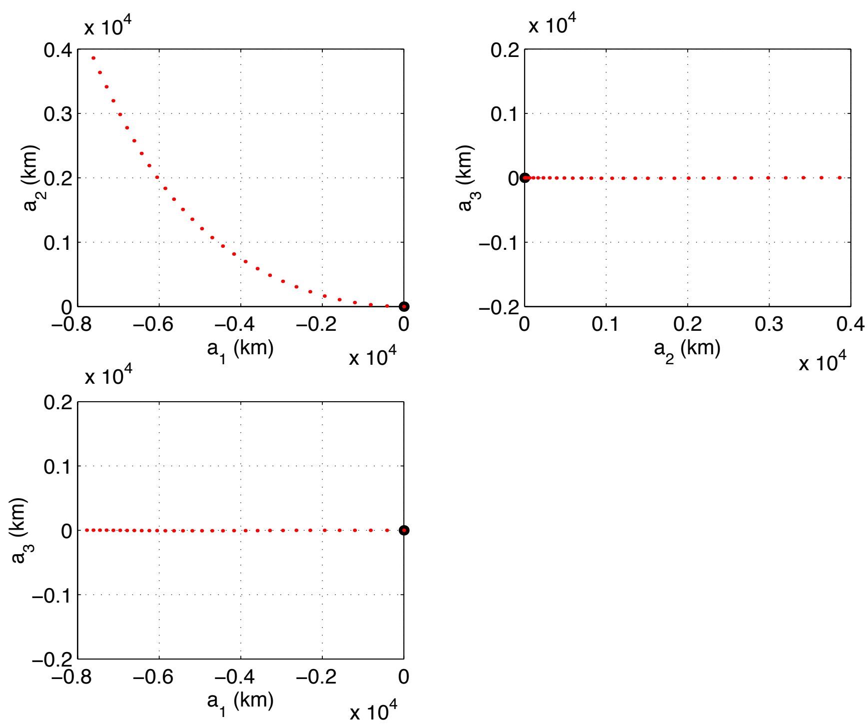

Figure 6: Spacecraft Motion In the Absence of Lunar Perturbation

CONCLUSION

A spacecraft, stationed on the Sun-Earth line near the transEarth Lagrange point $\mathcal{L}_2$ in order to obtain measurements of Earth's atmosphere as it occults the Sun, must possess a propulsion system capable of countering the lunar gravitational perturbation. Expressions are provided for the associated force per unit mass and corresponding velocity increment. A spacecraft permitted to move along the Sun-Earth line in response to the moon's influence needs less than half the amount of propellant required to negate completely the lunar disturbance. The expressions for lunar perturbing force and corresponding velocity increment are evaluated numerically, and simulations of motion give some indication of the efficacy of opposing the lunar perturbation by employing a propulsion system.

Although this work provides an adequate first order analysis of what the propulsion system must do to control the orbit of the spacecraft, a second order study provides a more accurate picture of propellant requirements and interesting details of spacecraft motion that result from meeting the stated constraints with optimal control. Just such an investigation is presented in Ref. [11].

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REFERENCES

[1] Farquhar, R. W., and Dunham, D. W., "Use of Libration-point Orbits for Space Observatories," *Observatories in Earth Orbit and Beyond*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1990, pp. 391–395.

[2] Farquhar, R. W., "The Role of the Sun-Earth Collinear Libration Points in Future Space Exploration," *Space Times*, Nov.–Dec., 2000, pp. 9–12.

[3] Wiscombe, W., Herman, J., and Valero, F., "L-1 and L-2 Observatories for Earth Science in the Post-2010 Era," International Geoscience and Remote Sensing Symposium, 2002, IEEE, IGARSS '02, Vol. 1, 2002, pp. 365–367.

[4] Breakwell, J. V., and Brown, J. V., "The 'Halo' Family of 3-Dimensional Periodic Orbits in the Earth-Moon Restricted 3-Body Problem," *Celestial Mechanics*, Vol. 20, Nov. 1979, pp. 389–404.

[5] Howell, K. C., and Breakwell, J. V., "Almost Rectilinear Halo Orbits," *Celestial Mechanics*, Vol. 32, 1984, pp. 29–52.

[6] Battin, R. H., *An Introduction to The Mathematics and Methods of Astrodynamics*, AIAA, New York, 1987.

[7] Howell, K. C., and Guzmán, J. J., “Spacecraft Trajectory Design in the Context of a Coherent Restricted Four-body Problem with Application to the MAP Mission,” IAF Paper 00-A.5.06, 51st International Astronautical Congress, Rio de Janeiro, Brazil, Oct. 2–6, 2000.

[8] *The Astronomical Almanac for the Year 1999*, Nautical Almanac Office, United States Naval Observatory, U.S. Government Printing Office.

[9] Farquhar, R. W., “The Moon’s Influence on the Location of the Sun-Earth Exterior Libration Point,” *Celestial Mechanics*, Vol. 2, July 1970, pp. 131–133.

[10] Bate, R. R., Mueller, D. D., and White, J. E., *Fundamentals of Astrodynamics*, Dover Publications, Inc., New York, 1971.

[11] Shen, H., Kumar, R. R., and Seywald, H., “Minimum-Fuel Periodic Orbits in the Vicinity of a Fixed Point on the Sun-Earth Line: The Planar Case,” AAS 04-247, 14th AAS/AIAA Space Flight Mechanics Meeting, Maui, Hawaii, Feb. 8–12, 2004.