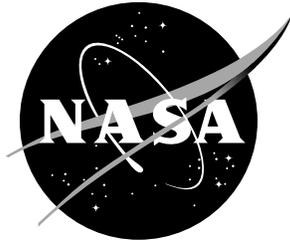


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Gain Weighted Eigenspace Assignment

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ABSTRACT

This report presents the development of the Gain Weighted Eigenspace Assignment methodology. This provides a designer with a systematic methodology for trading off eigenvector placement versus gain magnitudes, while still maintaining desired closed-loop eigenvalue locations. This is accomplished by forming a cost function composed of a scalar measure of error between desired and achievable eigenvectors and a scalar measure of gain magnitude, determining analytical expressions for the gradients, and solving for the optimal solution by numerical iteration. For this development the scalar measure of gain magnitude is chosen to be a weighted sum of the squares of all the individual elements of the feedback gain matrix. An example is presented to demonstrate the method. In this example, solutions yielding achievable eigenvectors close to the desired eigenvectors are obtained with significant reductions in gain magnitude compared to a solution obtained using a previously developed eigenspace (eigenstructure) assignment method.

1.0 INTRODUCTION

The Direct Eigenspace Assignment (DEA) method (Davidson 1986) is currently being used to design lateral-directional control laws for NASA's High Angle-of-Attack Research Vehicle (Davidson 1992). This method allows designers to shape the closed-loop response by choice of desired eigenvalues and eigenvectors. During this design effort DEA has been demonstrated to be a useful technique for aircraft control design. The control laws developed using DEA have demonstrated good performance, robustness, and flying qualities during piloted simulation. These control laws are scheduled for flight test at NASA Dryden Flight Research Center in 1994.

During the control law design effort, one limitation of the DEA method became apparent. Using DEA the designer has no direct control over augmentation gain magnitudes. Often it is not clear how to adjust the desired eigenspace in order to reduce individual undesirable gain magnitudes. To reduce undesirable gain magnitudes the designer must rely upon a strong physical insight into the dynamics or is forced to iterate on the design.

This report presents the development of an eigenspace (eigenstructure) assignment method that overcomes this limitation. This method, referred to as Gain Weighted Eigenspace Assignment (GWEA), allows a designer to place eigenvalues at desired locations and trade-off the achievement of desired eigenvectors versus feedback gain magnitudes.

This report is organized into four sections. Background information on how eigenvalues and eigenvectors influence a system's dynamic response and a review of the Direct Eigenspace Assignment methodology is presented in the following section. The development of the Gain Weighted Eigenspace Assignment methodology is presented in section 3. Concluding remarks are given in the final section.

2.0 BACKGROUND

This section presents a review of how eigenvalues and eigenvectors influence a system's dynamic response and a review of the Direct Eigenspace Assignment methodology.

Eigenvalues, Eigenvectors, and System Dynamic Response

The eigenvalues and eigenvectors of a system are related to its dynamic response in the following way. Given the observable and controllable linear time-invariant system

$$\dot{x} = Ax + Bu \quad (2.1a)$$

and output equation

$$y = Cx \quad (2.1b)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, and $y \in \mathbf{R}^l$.

The Laplace transform of equation (2.1a) is given by

$$sx(s) - x(0) = Ax(s) + Bu(s) \quad (2.2a)$$

$$x(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}Bu(s) \quad (2.2b)$$

Solution of equation (2.1a) is given by taking the inverse Laplace Transform of equation (2.2b)

$$x(t) = \mathcal{L}^{-1}\{[sI - A]^{-1}\}x(0) + \mathcal{L}^{-1}\{[sI - A]^{-1}Bu(s)\} \quad (2.3)$$

Noting that

$$\mathcal{L}^{-1}\{[sI - A]^{-1}\} = e^{At} \quad (2.4)$$

the solution of (2.3) is (Brogan 1974)

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \quad (2.5)$$

and system outputs are

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \quad (2.6)$$

The system dynamic matrix, A , can be represented by

$$A = V \Lambda V^{-1} = V \Lambda L \quad (2.7)$$

where V is a matrix of system eigenvectors, L is the inverse eigenvector matrix, and Λ is a diagonal matrix of system eigenvalues. Given this result, e^{At} can be expressed by

$$e^{At} = V e^{\Lambda t} L = \sum_{j=1}^n v_j e^{\lambda_j t} l_j \quad (2.8)$$

where λ_j is the j^{th} system eigenvalue, v_j is the j^{th} column of V (j^{th} eigenvector of A), and l_j is the j^{th} row of L (j^{th} left eigenvector of A). Equation (2.6) can then be expressed as

$$y(t) = \sum_{j=1}^n C v_j e^{\lambda_j(t)} l_j x(0) + \sum_{j=1}^n C v_j \int_0^t e^{\lambda_j(t-\tau)} l_j B u(\tau) d\tau \quad (2.9)$$

Noting that

$$B u(t) = \sum_{k=1}^m b_k u_k(t) \quad (2.10)$$

where b_k is the k^{th} column of B and u_k is the k^{th} system input, the system outputs due to initial conditions and input u_k is given by

$$y(t) = \sum_{j=1}^n C v_j e^{\lambda_j(t)} l_j x(0) + \sum_{j=1}^n \sum_{k=1}^m C v_j l_j b_k \int_0^t e^{\lambda_j(t-\tau)} u_k(\tau) d\tau \quad (2.11)$$

The i^{th} system output is given by

$$y_i(t) = \sum_{j=1}^n c_i v_j e^{\lambda_j(t)} l_j x(0) + \sum_{j=1}^n \sum_{k=1}^m c_i v_j l_j b_k \int_0^t e^{\lambda_j(t-\tau)} u_k(\tau) d\tau \quad (2.12)$$

where c_i is the i^{th} row of C . In the case of initial conditions equal to zero, the i^{th} output is given by

$$y_i(t) = \sum_{j=1}^n \sum_{k=1}^m R_{ijk} \int_0^t e^{\lambda_j(t-\tau)} u_k(\tau) d\tau \quad (2.13)$$

where $R_{ijk} = c_i v_j l_j b_k$. In this expression, R_{ijk} is the modal residue for output i , associated with eigenvalue λ_j , and due to input k .

Given an impulsive input in the k^{th} input, equation (2.13) reduces to

$$y_i(t) = \sum_{j=1}^n \sum_{k=1}^m R_{ijk} e^{\lambda_j(t)} \quad (2.14)$$

As this expression shows, for an impulsive input, a system's dynamics are dependent on both its eigenvalues and its eigenvectors. The eigenvalues determine the natural frequency and damping of each mode. The eigenvectors determine the residues. The residues are an indicator of how much each mode of the system contributes to a given output.

Direct Eigenspace Assignment Methodology

One possible approach to the aircraft control synthesis problem would be to synthesize a control system that would control both the eigenvalue locations and the residue magnitudes associated with undesirable modes in certain outputs. Since the residues are a function of the system's eigenvectors this naturally leads to a control synthesis technique that involves achieving some desired eigenspace in the closed-loop system (Moore 1976; Srinathkumar 1978; Cunningham 1980; Andry 1983; Smith 1990). Davidson and Schmidt (Davidson 1986) explored this approach by using Direct Eigenspace Assignment (DEA) to synthesize flight control systems for flexible aircraft. DEA is a control synthesis technique for directly determining measurement feedback control gains that will yield an achievable eigenspace in the closed-loop system. For a system that is observable and controllable and has n states, m controls, and l measurements; DEA will determine a gain matrix that will place l eigenvalues to desired locations and m elements of their associated eigenvectors to desired values [†]. If it is desired to place more than m elements of the associated l eigenvectors, DEA yields eigenvectors in the closed-loop system that are as close as possible in a least squares sense to desired eigenvectors. The following section will present the development of the DEA synthesis technique. A more detailed development can be found in Davidson 1986.

Direct Eigenspace Assignment Formulation

Given the observable, controllable system

$$\dot{x} = Ax + Bu \quad (2.15a)$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$, with system measurements given by

$$z = Mx + Nu \quad (2.15b)$$

where $z \in \mathbf{R}^l$.

The total control input is the sum of the augmentation input u_c and pilot's input u_p

$$u = u_p + u_c \quad (2.16)$$

The measurement feedback control law is

[†] This assumes $l > m$. For a general statement and proof of this property the reader is referred to Srinathkumar 1978.

$$u_c = Gz \quad (2.17)$$

Solving for u as a function of the system states and pilot's input yields

$$u = [I_m - GN]^{-1} GMx + [I_m - GN]^{-1} u_p \quad (2.18)$$

The system augmented with the control law is given by

$$\dot{x} = (A + B[I_m - GN]^{-1} GM)x + B[I_m - GN]^{-1} u_p \quad (2.19)$$

The spectral decomposition of the closed-loop system is given by

$$(A + B[I_m - GN]^{-1} GM)v_i = \lambda_i v_i \quad (2.20)$$

for $i = 1, \dots, n$ where λ_i is the i^{th} system eigenvalue and v_i is the associated i^{th} system eigenvector. Let w_i be defined by

$$w_i \equiv [I_m - GN]^{-1} GM v_i \quad (2.21)$$

Substituting this result into equation (2.20) and solving for v_i one obtains

$$v_i = [\lambda_i I_n - A]^{-1} B w_i \quad (2.22)$$

This equation describes the achievable i^{th} eigenvector of the closed-loop system as a function of the eigenvalue λ_i and w_i . By examining this equation, one can see that the number of control variables (m) determines the dimension of the subspace in which the achievable eigenvectors must reside.

Values of w_i that yield an achievable eigenspace that is as close as possible in a least squares sense to a desired eigenspace can be determined by defining a cost function associated with the i^{th} mode of the system

$$J_i = \frac{1}{2} (v_{a_i} - v_{d_i})^H Q_{d_i} (v_{a_i} - v_{d_i}) \quad (2.23)$$

for $i = 1, \dots, l$ where v_{a_i} is the i^{th} achievable eigenvector associated with eigenvalue λ_i , v_{d_i} is the i^{th} desired eigenvector, and Q_{d_i} is an n -by- n symmetric positive semi-definite weighting matrix on eigenvector elements † . This cost function represents the error between the achievable eigenvector and the desired eigenvector weighted by the matrix Q_{d_i} .

† Superscript H denotes complex conjugate transpose (Strang 1980).

Values of w_i that minimize J_i are determined by substituting (2.22) into the cost function for v_{a_i} , taking the gradient of J_i with respect to w_i , setting this result equal to zero, and solving for w_i . This yields

$$w_i = [L_i^H Q_{d_i} L_i]^{-1} L_i^H Q_{d_i} v_{d_i} \quad (2.24)$$

where

$$L_i = [\lambda_{d_i} I_n - A]^{-1} B \quad (2.25)$$

and λ_{d_i} is the i^{th} desired eigenvalue of the closed-loop system. Note in this development λ_{d_i} cannot belong to the spectrum of A .

By concatenating the individual w_i 's column-wise to form W and v_{a_i} 's column-wise to form V_a , equation (2.21) can be expressed in matrix form by

$$W = [I_m - GN]^{-1} GM V_a \quad (2.26)$$

The feedback gain matrix that yields the desired closed-loop eigenvalues and achievable eigenvectors is given by

$$G = W[MV_a + NW]^{-1} \quad (2.27)$$

Design Algorithm

A feedback gain matrix that yields a desired closed-loop eigenspace is determined in the following way:

- 1) Select desired eigenvalues λ_{d_i} , desired eigenvectors v_{d_i} , and desired eigenvector weighting matrices Q_{d_i} .
- 2) Calculate w_i 's using equation (2.24) and concatenate these column-wise to form W .
- 3) Calculate achievable eigenvectors v_{a_i} 's using equation (2.22) and concatenate these column-wise to form V_a .
- 4) The feedback gain matrix G is then calculated using equation (2.27).

Example

An example will be presented to demonstrate the method. The design model is the lateral/directional dynamics of a high performance aircraft at low angle-of-attack. The model is based on a steady-state one g trim flight condition of forward cruise speed equaling 598 feet/second at 25,000 feet. It includes the four standard lateral-directional

rigid-body degrees of freedom. The design goal is to improve the flying qualities by placing eigenvalues at level one locations and choosing eigenvectors to decouple the roll and dutch roll modes.

The model is as follows:

$$\dot{x} = Ax + Bu \text{ (system dynamics)} \quad (2.28a)$$

$$z = Mx + Nu \text{ (system measurements)} \quad (2.28b)$$

$$u = u_p + u_c \text{ (total control input)} \quad (2.28c)$$

$$u_c = Gz \text{ (feedback control law)} \quad (2.28d)$$

$$u_p = G_p \delta_p \text{ (pilot control input)} \quad (2.28e)$$

The system states are:

$$x^T = [\beta \ p_s \ r_s \ \phi]^T \quad (2.29)$$

where

β = sideslip angle (rad)

p_s = stability axis roll rate (rad/sec)

r_s = stability axis yaw rate (rad/sec)

ϕ = bank angle (rad)

The system controls are:

$$u^T = [a_{roll} \ a_{yaw}]^T \quad (2.30)$$

where

a_{roll} = stability axis roll acceleration (rad/sec²)

a_{yaw} = stability axis yaw acceleration (rad/sec²)

The measurements considered for feedback are:

$$z^T = [p_s \ r_s \ a_y \ \dot{\beta}]^T \quad (2.31)$$

where

p_s = stability axis roll rate (rad/sec)

r_s = stability axis yaw rate (rad/sec)

a_y = lateral acceleration at the c.g.(g's)

$\dot{\beta}$ = sideslip rate (rad/sec)

The open-loop system matrices and open-loop eigenvalues and eigenvectors are given in Figure 1. In this Figure, the open-loop eigenvectors have been scaled to allow comparison with the desired closed-loop eigenvectors.

For this example, there are four states, two controls, and four measurements; therefore using DEA one can place four eigenvalues to desired locations and exactly place two elements of each associated eigenvector. The desired eigenvalues are chosen to yield good flying qualities. The desired roll and dutch roll eigenvectors are chosen to decouple the roll and dutch roll modes in the roll rate and sideslip responses. The desired eigenvalues and eigenvectors are given in Table 1. In this Table, an x denotes eigenvector elements that are not weighted in the cost function. Therefore, the desired value for these elements is taken as arbitrary. Diagonal weighting matrices were used. Desired elements are weighted unity and other elements were weighted zero.

The gain matrix to obtain the achievable eigenspace in the closed-loop system and closed-loop eigenspace is given in Figure 2. In this Figure, the closed-loop eigenvectors have been scaled to allow comparison with the desired eigenvectors.

The closed loop eigenvalues have been placed at desired locations. As can be seen by examining the desired and achievable eigenvector elements, all the desired elements were obtained.

DEA Conclusions

DEA is a control synthesis technique for directly determining measurement feedback gains that will yield an achievable closed-loop eigenspace. For an observable controllable system that has n states, m controls, and l measurements one can determine a gain matrix that will place l eigenvalues to the desired locations and their associated eigenvectors as close as possible in a least squares sense to desired eigenvectors.

Using DEA the designer has no direct control over augmentation gain magnitudes. Often it is not clear how to adjust the desired eigenspace in order to reduce individual undesirable gain magnitudes. To reduce undesirable gain magnitudes the designer must rely upon a strong physical insight into the dynamics or is forced to iterate on the design.

The next section presents an eigenspace assignment methodology that overcomes this limitation.

3.0 GAIN WEIGHTED EIGENSPACE ASSIGNMENT

This section presents the development of the Gain Weighted Eigenspace Assignment (GWEA) methodology. This method allows a designer to place l eigenvalues at desired locations and trade-off the achievement of desired eigenvectors versus feedback gain magnitudes.

The GWEA formulation builds upon a matrix formulation of DEA. The matrix DEA formulation is presented first followed by the GWEA formulation. The following development assumes complex matrices have been converted to real Jordan form (Brogan 1974).

Matrix DEA Formulation

Given the observable, controllable system

$$\dot{x} = Ax + Bu \quad (3.1a)$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$, with system measurements available for feedback given by

$$z = Mx + Nu \quad (3.1b)$$

where $z \in \mathbf{R}^l$.

The total control input is the sum of the augmentation input u_c and pilot's input u_p

$$u = u_p + u_c \quad (3.2)$$

The measurement feedback control law is

$$u_c = Gz \quad (3.3)$$

Solving for u as a function of the system states and pilot's input yields

$$u = [I_m - GN]^{-1} GMx + [I_m - GN]^{-1} u_p \quad (3.4)$$

The system augmented with the control law is given by

$$\dot{x} = (A + B[I_m - GN]^{-1} GM)x + B[I_m - GN]^{-1} u_p \quad (3.5)$$

The spectral decomposition of the closed-loop system is given by

$$(A + B[I_m - GN]^{-1} GM)v_i = \lambda_i v_i \quad (3.6)$$

for $i = 1, \dots, n$ where λ_i is the i^{th} system eigenvalue and v_i is the associated i^{th} system eigenvector. In matrix form this is given by

$$(A + B(I_m - GN)^{-1} GM)V = V\Lambda \quad (3.7)$$

Let W be defined by

$$W \equiv (I_m - GN)^{-1} GMV \quad (3.8)$$

Substituting this equation into (3.7) and taking the vector value of each term yields

$$\text{vec}(AV) + \text{vec}(BW) = \text{vec}(V\Lambda) \quad (3.9)$$

where $\text{vec}(X)$ is a vector valued function denoting a vector description of the matrix X (Graham 1981). Applying properties and rules for Kronecker products, equation (3.9) can be rewritten as

$$(I_l \otimes A)\text{vec} V + (I_l \otimes B)\text{vec} W = (\Lambda^T \otimes I_n)\text{vec} V \quad (3.10)$$

where \otimes denotes Kronecker product. Some definitions and rules for Kronecker products used in this development are given in the appendix.

Solving for $\text{vec} V$ as a function of Λ and $\text{vec} W$ one obtains

$$\text{vec} V = [(\Lambda^T \otimes I_n) - (I_l \otimes A)]^{-1} (I_l \otimes B)\text{vec} W \quad (3.11)$$

or

$$\text{vec} V = A_D \text{vec} W \quad (3.11b)$$

where

$$A_D = [(\Lambda^T \otimes I_n) - (I_l \otimes A)]^{-1} (I_l \otimes B) \quad (3.12)$$

This equation (3.11) describes the achievable eigenvectors of the closed-loop system as a function of the desired closed-loop eigenvalues and $\text{vec} W$. If one could calculate a $\text{vec} W$ that would make the achievable eigenvectors as close as possible in a least squares sense to some desired eigenvectors, it could be used to determine a gain matrix that would yield this eigenspace. One way this can be done is to define a cost function

$$J_e = \text{vec}^T (V_a - V_d) Q_d \text{vec} (V_a - V_d) = e^T Q_d e \quad (3.13)$$

where

V_a = achievable system eigenvectors

V_d = desired system eigenvectors

Q_d = symmetric positive semi-definite weighting matrix (eigenvector weighting matrix)

$e = \text{vec}(V_a - V_d)$.

This cost function represents the error between the achievable eigenvectors and the desired eigenvectors weighted by the matrix Q_d .

The value of $\text{vec} W$ that will minimize J_e can be obtained by taking the partial of J_e with respect to $\text{vec} W$. This partial can be determined in the following way. Substituting equation (3.11) into equation (3.13) for V_a yields J_e as a function of $\text{vec} W$. By applying the vector chain rule property (Graham 1981), the partial of J_e with respect to $\text{vec} W$ is given by

$$\frac{\partial J_e}{\partial \text{vec}W} = \left(\frac{\partial e}{\partial \text{vec}W} \right) \left(\frac{\partial J_e}{\partial e} \right) = \left(\frac{\partial e}{\partial \text{vec}W} \right) \left(\frac{\partial}{\partial e} (e^T Q_d e) \right) \quad (3.14)$$

Differentiating each term yields

$$\frac{\partial}{\partial e} (e^T Q_d e) = Q_d e + Q_d^T e = 2Q_d e \quad (3.15)$$

and

$$\frac{\partial e}{\partial \text{vec}W} = \frac{\partial}{\partial \text{vec}W} (\text{vec}(V_a - V_d)) = \frac{\partial}{\partial \text{vec}W} (A_D \text{vec}W - \text{vec}V_d) = A_D^T \quad (3.16)$$

Therefore, the partial of J_e with respect to $\text{vec}W$ is given by

$$\frac{\partial J_e}{\partial \text{vec}W} = 2A_D^T Q_d e = 2A_D^T Q_d (\text{vec}V_a - \text{vec}V_d) \quad (3.17)$$

In this case, a closed form solution for $\text{vec}W$ exists. This is obtained by setting equation (3.17) equal to zero and solving for $\text{vec}W$. This yields

$$\text{vec}W = [A_D^T Q_d A_D]^{-1} A_D^T Q_d \text{vec}V_d \quad (3.18)$$

where

$$A_D = [(\Lambda_d^T \otimes I_n) - (I_l \otimes A)]^{-1} (I_l \otimes B)$$

and Λ_d is a block diagonal matrix of desired closed-loop eigenvalues. Note in this development the desired closed-loop eigenvalues cannot belong to the spectrum of A .

The gain matrix that will yield the desired eigenvalues and achievable eigenvectors is obtained by solving equation (3.8) for G .

$$G = W(MV_a + NW)^{-1} \quad (3.19)$$

By noting that

$$V_a = (\text{vec}^T I_l \otimes I_n)(I_l \otimes A_D)(I_2 \otimes W)(I_l \otimes \text{vec}I_l) \quad (3.20)$$

the feedback gains can be expressed as a function of the desired closed-loop eigenvalues and W by

$$G = W(MK_2(I_2 \otimes W)K_3 + NW)^{-1} \quad (3.21)$$

where

$$K_2 = (\text{vec}^T I_l \otimes I_n)(I_l \otimes A_D)$$

$$K_3 = (I_l \otimes \text{vec} I_l)$$

Gain Weighted Eigenspace Assignment Formulation

The Gain Weighted Eigenspace Assignment formulation extends the Direct Eigenspace Assignment formulation to allow trading off eigenvector placement versus gain magnitudes, while still maintaining desired closed-loop eigenvalue locations. This is accomplished by appending a scalar measure of gain magnitude that is a function of $\text{vec}W$ to the cost function given in equation (3.13), determining partials with respect to $\text{vec}W$, and solving for the optimal solution by numerical iteration (Fletcher 1963).

For this development, the scalar measure of gain magnitude is chosen to be a weighted sum of the squares of all the individual elements of the feedback gain matrix. To maximize design flexibility, the gain magnitude term is formulated to allow weighting individual elements of the feedback gain matrix. A gain magnitude cost function that allows this can be formed in terms of the vector value of G .

The vector value of G as a function of $\text{vec}W$ is obtained by taking the vector value of equation (3.19) and noting that

$$V_a = (\text{vec}^T I_l \otimes I_n)(I_l \otimes \text{vec} V_a) \quad (3.22)$$

$$W = (\text{vec}^T I_l \otimes I_m)(I_l \otimes \text{vec} W) \quad (3.23)$$

thus yielding

$$\begin{aligned} \text{vec} G &= [(MV_a + NW)^{-T} \otimes I_m] \text{vec} W \\ &= [(M(\text{vec}^T I_l \otimes I_n)(I_l \otimes A_D \text{vec} W) + N(\text{vec}^T I_l \otimes I_m)(I_l \otimes \text{vec} W))^{-T} \otimes I_m] \text{vec} W \end{aligned} \quad (3.24)$$

The gain magnitude cost function is given by

$$J_g = \text{vec}^T(G) Q_g \text{vec}(G) = g^T Q_g g \quad (3.25)$$

where

Q_g = symmetric positive semi-definite weighting matrix (gain weighting matrix)
 $g = \text{vec}(G)$.

This cost function represents the sum of the square of the individual feedback gains, each weighted by an element of the diagonal of the matrix Q_g . The value of $\text{vec}W$ that yields minimum gain magnitudes while achieving the desired closed-loop eigenvalues is determined by minimizing this cost function.

Trade-offs between achievement of desired system eigenvectors and minimizing gain magnitudes can be made by forming the composite cost function

$$J = \rho_e J_e + \rho_g J_g \quad (3.26)$$

where ρ_e and ρ_g are scalar positive cost function weights on the eigenvector placement error and gain magnitudes, respectively.

Because eigenvectors can be scaled by an arbitrary constant, a unique solution to this cost function (3.26) does not exist when ρ_e is zero (or in practice when ρ_e is small compared to ρ_g). To ensure a unique solution for all values of ρ_e and ρ_g it is necessary to constrain the eigenvectors to be unique. This can be accomplished by forcing one element of each eigenvector to be a specific reference value. To be consistent with the eigenvector error term J_e in equation (3.26), this specific value is chosen to be an element of each desired eigenvector. This equality constraint can be expressed in the form of a penalty function (Bryson 1975) as

$$J_r = \text{vec}^T (V_a - V_d) Q_r \text{vec} (V_a - V_d) = e^T Q_r e \quad (3.27)$$

where

Q_r = symmetric positive semi-definite weighting matrix (reference weighting matrix)
 $e = \text{vec} (V_a - V_d)$.

This penalty function will be referred to as an eigenvector reference constraint. It represents the error between an element of each achievable eigenvector and the corresponding reference element of the desired eigenvector. The weighting matrix Q_r is chosen to weight one element of each desired eigenvector.

Appending this penalty function to the cost function (3.26) yields

$$J = \rho_e e^T Q_d e + \rho_g g^T Q_g g + \rho_r e^T Q_r e \quad (3.28)$$

where ρ_e , ρ_g , and ρ_r are scalar positive cost function weights on the eigenvector placement error, gain magnitudes, and the eigenvector reference constraint; respectively. With this cost function, trade-offs between achievement of desired system eigenvectors and minimizing gain magnitudes can be made by choice of values of ρ_g and ρ_e . To ensure a unique solution the eigenvector reference constraint weighting ρ_r should be chosen to be very large compared to the values of ρ_g and ρ_e .

Partials of J with respect to $\text{vec}W$

The value of $\text{vec}W$ that will minimize J can be obtained by determining the gradient of J with respect to $\text{vec}W$. The partial of J with respect to $\text{vec}W$, is given by

$$\frac{\partial J}{\partial \text{vec}W} = \rho_e \frac{\partial J_e}{\partial \text{vec}W} + \rho_g \frac{\partial J_g}{\partial \text{vec}W} + \rho_r \frac{\partial J_r}{\partial \text{vec}W} \quad (3.29)$$

The partial of J_e with respect to $vecW$ is given by equation (3.17).

The partial of J_g with respect to $vecW$ can be determined in the following way. Applying the vector chain rule yields

$$\frac{\partial J_g}{\partial vecW} = \left(\frac{\partial g}{\partial vecW} \right) \left(\frac{\partial J_g}{\partial g} \right) = \left(\frac{\partial g}{\partial vecW} \right) \left(\frac{\partial}{\partial g} (g^T Q_g g) \right) \quad (3.30)$$

Differentiating each term yields

$$\frac{\partial}{\partial g} (g^T Q_g g) = Q_g g + Q_g^T g = 2Q_g g \quad (3.31)$$

and

$$\frac{\partial g}{\partial vecW} = \frac{\partial vecG}{\partial vecW} = \left[vec \frac{\partial G}{\partial w_{11}} : vec \frac{\partial G}{\partial w_{21}} : \dots : vec \frac{\partial G}{\partial w_{ml}} \right]^T \quad (3.32)$$

where w_{ij} is the (i,j)th element of W .

By applying the matrix chain rule, the partial of G with respect to w_{ij} is given by

$$\begin{aligned} \frac{\partial G}{\partial w_{ij}} &= \frac{\partial}{\partial w_{ij}} \left(W(MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} \right) \\ &= \left(\frac{\partial W}{\partial w_{ij}} \right) (MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} + W \left(\frac{\partial}{\partial w_{ij}} (MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} \right) \end{aligned} \quad (3.33)$$

The first partial with respect to w_{ij} in this equation is given by

$$\left(\frac{\partial W}{\partial w_{ij}} \right) = U_{ij} \quad (3.34)$$

where U_{ij} is a matrix of order m -by- l which has unity in the (i,j)th position and all other elements are zero.

The second partial in equation (3.33) is given by

$$\begin{aligned} & \left(\frac{\partial}{\partial w_{ij}} (MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} \right) = \\ & -(MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} \left(\frac{\partial}{\partial w_{ij}} (MK_2(I_{l^2} \otimes W)K_3 + NW) \right) (MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} \end{aligned} \quad (3.35)$$

where

$$\begin{aligned} \frac{\partial}{\partial w_{ij}} (MK_2(I_{l^2} \otimes W)K_3 + NW) &= MK_2 \left(\frac{\partial}{\partial w_{ij}} (I_{l^2} \otimes W) \right) K_3 + N \frac{\partial W}{\partial w_{ij}} \\ &= MK_2(I_{l^2} \otimes \frac{\partial W}{\partial w_{ij}})K_3 + N \frac{\partial W}{\partial w_{ij}} \\ &= MK_2(I_{l^2} \otimes U_{ij})K_3 + NU_{ij} \end{aligned} \quad (3.36)$$

Therefore, the partial of G with respect to w_{ij} is

$$\begin{aligned} \frac{\partial G}{\partial w_{ij}} &= U_{ij} (MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} \\ &\quad - W(MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} (MK_2(I_{l^2} \otimes U_{ij})K_3 + NU_{ij}) (MK_2(I_{l^2} \otimes W)K_3 + NW)^{-1} \\ &= U_{ij} (MV_a + NW)^{-1} - G(MK_2(I_l \otimes \text{vec} U_{ij}) + NU_{ij}) (MV_a + NW)^{-1} \\ &= (I_m - GN)U_{ij} (MV_a + NW)^{-1} - GMK_2(I_l \otimes \text{vec} U_{ij}) (MV_a + NW)^{-1} \end{aligned} \quad (3.37)$$

Taking the vector value of both sides of this equation yields

$$\text{vec} \frac{\partial G}{\partial w_{ij}} = \left((MV_a + NW)^{-T} \otimes (I_m - GN) \right) \text{vec} U_{ij} - \left((MV_a + NW)^{-T} \otimes GMK_2 \right) \text{vec} (I_l \otimes \text{vec} U_{ij}) \quad (3.38)$$

Substituting equation (3.38) into equation (3.32) yields

$$\begin{aligned}
\frac{\partial g}{\partial \text{vec}W} &= \left[\left((MV_a + NW)^{-T} \otimes (I_m - GN) \right) \left[\text{vec}U_{11} : \text{vec}U_{21} : \dots : \text{vec}U_{m1} \right] \right]^T \\
&\quad - \left[\left((MV_a + NW)^{-T} \otimes GMK_2 \right) \left[\text{vec}(I_l \otimes \text{vec}U_{11}) : \dots : \text{vec}(I_l \otimes \text{vec}U_{m1}) \right] \right]^T \\
&= \left[\text{vec}U_{11} : \text{vec}U_{21} : \dots : \text{vec}U_{m1} \right]^T \left((MV_a + NW)^{-1} \otimes (I_m - GN)^T \right) \\
&\quad - \left[\text{vec}(I_l \otimes \text{vec}U_{11}) : \dots : \text{vec}(I_l \otimes \text{vec}U_{m1}) \right]^T \left((MV_a + NW)^{-1} \otimes (GMK_2)^T \right)
\end{aligned} \tag{3.39}$$

By noting that

$$I_{ml} = \left[\text{vec}U_{11} : \text{vec}U_{21} : \dots : \text{vec}U_{m1} \right] \tag{3.40}$$

and defining

$$\bar{U} = \left[\text{vec}(I_l \otimes \text{vec}U_{11}) : \text{vec}(I_l \otimes \text{vec}U_{21}) : \dots : \text{vec}(I_l \otimes \text{vec}U_{m1}) \right] \tag{3.41}$$

the partial of g with respect to $\text{vec}W$ is

$$\frac{\partial g}{\partial \text{vec}W} = \left((MV_a + NW)^{-1} \otimes (I_m - GN)^T \right) - \bar{U}^T \left((MV_a + NW)^{-1} \otimes (GMK_2)^T \right) \tag{3.42}$$

Therefore, the partial of J_g with respect to $\text{vec}W$ is

$$\frac{\partial J_g}{\partial \text{vec}W} = 2 \left[\left((MV_a + NW)^{-1} \otimes (I_m - GN)^T \right) - \bar{U}^T \left((MV_a + NW)^{-1} \otimes (GMK_2)^T \right) \right] Q_g g \tag{3.43}$$

The partial of J_r is determined in a manner similar to the partial of J_e . The partial of J_r with respect to $\text{vec}W$ is given by

$$\frac{\partial J_r}{\partial \text{vec}W} = 2A_D^T Q_r e = 2A_D^T Q_r (\text{vec}V_a - \text{vec}V_d) \tag{3.44}$$

Design Algorithm

A feedback gain matrix can be calculated using the GWEA algorithm in the following way:

1) Select desired eigenvalues Λ_d , desired eigenvectors V_d , and desired eigenvector weighting matrix Q_d .

- 2) Select gain weighting matrix Q_g and reference weighting matrix Q_r .
- 3) Select cost function weights (see (3.28)) on eigenvector error ρ_e and gain magnitude ρ_g .
- 4) Set cost function reference constraint weighting $\rho_r = 100.0 * \max(\rho_e, \rho_g)$.
- 5) Calculate W that minimizes cost function (3.28) using analytic gradients in (3.17), (3.43), and (3.44) along with numerical optimization techniques in Fletcher 1963.
- 6) Achievable eigenvectors V_a can be calculated using equation (3.20).
- 7) The feedback gain matrix G is then calculated using equation (3.24) or (3.19).
- 8) Adjust cost function weights ρ_g and ρ_e as necessary to achieve desired design trade-offs.

Example

An example will be presented to demonstrate the method. This example uses the same model presented in Section 2; the lateral/directional dynamics of a high performance aircraft at low angle-of-attack. The open-loop system matrices and open-loop eigenvalues and eigenvectors are given in Figure 1.

The desired eigenvalues and eigenvectors are chosen to be the same as in the previous example. The desired eigenvalues are chosen to yield good flying qualities. The desired roll and dutch roll eigenvectors are chosen to decouple the roll and dutch roll modes in the roll rate and sideslip responses. The desired eigenvalues, desired eigenvectors, and eigenvector weighting matrix are given in Table 1. The gain weighting matrix Q_g was set to identity. The desired eigenvector elements weighted in the reference weighting matrix Q_r are given in Table 2. The reference constraint weighting ρ_r was set to $100.0 * \max(\rho_e, \rho_g)$.

The algorithm was implemented in MATLAB [†] and executed on a SUN SPARC 10 ^{††} workstation. Designs were determined for ten values of ρ_g / ρ_e . Solutions for each design were obtained in less than five minutes and within 50 iterations. Values of ρ_g / ρ_e , J_e , and J_g for these designs are given in Table 3. A plot of J_e versus the square root of J_g is given in Figure 3. Gain matrices, closed-loop system matrices, and closed-loop eigenspaces for four values of ρ_g / ρ_e (0.0, 0.1, 100.0, 1.0e05) are given in Figures 4 - 7. In these Figures, the closed-loop eigenvectors have been scaled to allow comparison with the desired eigenvectors.

Cost function weighting $\rho_g / \rho_e = 0.0$ yields the DEA solution - four eigenvalues at desired locations and their associated eigenvectors are as close as possible in a least squares sense to the desired eigenvectors. The solution for this weighting is the same as the DEA solution presented in Figure 2. The RMS gain magnitude for this design is 5.73.

Cost function weighting $\rho_g / \rho_e = 0.1$ yields four eigenvalues at desired locations and their associated eigenvectors close to the desired eigenvectors with a RMS gain magnitude of 4.08. This is a RMS gain magnitude reduction of approximately 28% compared to the DEA solution.

[†] MATLAB is a registered trademark of The MathWorks, Inc.

^{††} SUN SPARC 10 is a registered trademark of Sun Microsystems, Inc.

Cost function weighting $\rho_g / \rho_e = 1.0e05$ yields four eigenvalues at desired locations and eigenvectors that minimize feedback gain magnitudes (as defined by equation (3.25)). The RMS gain magnitude for this design is 2.13. This is a RMS gain magnitude reduction of approximately 63% compared to the DEA solution.

4.0 CONCLUDING REMARKS

This report has presented the development of the Gain Weighted Eigenspace Assignment methodology. This provides a designer with a systematic methodology for trading off eigenvector placement versus gain magnitudes, while still maintaining desired closed-loop eigenvalue locations. This was accomplished by forming a cost function composed of a scalar measure of error between desired and achievable eigenvectors and a scalar measure of gain magnitude, determining analytical expressions for the gradients, and solving for the optimal solution by numerical iteration. For this development the scalar measure of gain magnitude was chosen to be a weighted sum of the squares of all the individual elements of the feedback gain matrix. To achieve a solution it was necessary to constrain the system eigenvectors to be unique. This was accomplished by appending a penalty function to the cost function.

An example was presented to demonstrate the method. In this example it was shown that cost function weighting $\rho_g / \rho_e = 0.0$ yielded the Direct Eigenspace Assignment solution - closed-loop eigenvalues at desired locations and their associated eigenvectors are as close as possible in a least squares sense to desired eigenvectors. As the cost function weighting ρ_g / ρ_e tended towards infinity the solution yielded closed-loop eigenvalues at desired locations and eigenvectors that minimized feedback gain magnitudes. Solutions yielding achievable eigenvectors close to the desired eigenvectors could be obtained with significant reductions in gain magnitude compared to the Direct Eigenspace Assignment solution.

5.0 REFERENCES

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APPENDIX

This appendix presents some definitions and rules used in the development of the Gain Weighted Eigenspace Assignment methodology. For more information and proofs of these properties the reader is referred to Graham 1981.

Definition of the VEC Operator

Given a matrix A of order m -by- n , a vector valued function of matrix A , denoted by $vecA$, is defined by

$$vecA = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{mn} \end{bmatrix}$$

where a_{ij} denotes the (i,j) th element of the matrix A . The vector $vecA$ is of order mn -by-1.

Definition of the Kronecker Product

Given a matrix A of order m -by- n and a matrix B of order r -by- p , the Kronecker product of the two matrices, denoted by $A \otimes B$, is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

where a_{ij} denotes the (i,j) th element of the matrix A . The matrix $A \otimes B$ is of order mr -by- np .

Some Rules for Kronecker Products

$$A \otimes (B + C) = (A \otimes B) + (A \otimes C)$$

$$(A + B) \otimes C = (A \otimes C) + (B \otimes C)$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD \quad (\text{provided matrix dimensions are compatible})$$

$$(A \otimes B)^T = A^T \otimes B^T$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \quad (\text{subject to existence of matrix inverses})$$

$$vec(A Y B) = (B^T \otimes A) vecY$$

Open-loop System Matrices

A =

-0.1305	0.0003	-0.9978	0.0537
-11.1989	-1.5271	0.6757	0
2.9944	0.1152	-0.1529	0
0	1.0000	0	0

B =

-0.0033	-0.0319
1.1179	-0.1941
0.0096	1.3527
0	0

M =

0	1.0000	0	0
0	0	1.0000	0
-1.2576	0.0535	-0.0462	0
-0.1198	0.0003	-0.9992	0.0538

N =

0	0
0	0
-0.0614	-0.0669
-0.0033	-0.0319

Open-loop Eigenvalue Matrix

-0.2072 + 1.6584i	0	0	0
0	-0.2072 - 1.6584i	0	0
0	0	-1.4004	0
0	0	0	0.0043

Open-loop Eigenvector Matrix

1.0000	1.0000	-0.0147	0.0026
-3.6375 + 3.7717i	-3.6375 - 3.7717i	1.0000	0.0043
0.2110 - 1.5598i	0.2110 + 1.5598i	-0.0569	0.0535
2.5092 + 1.8800i	2.5092 - 1.8800i	-0.7141	1.0000

Figure 1 - Open-Loop System Matrices and Eigenspace.

Gain Matrix and Gain Magnitude

G =

-0.3096	-0.8200	-5.1642	-0.6697
0.0230	0.1138	-0.6396	2.1203

gainmag =

5.73

Closed-loop System Matrix

A+B*inv(Im-G*N)*G*M =

-0.1916	0.0044	-0.9324	0.0501
-0.0519	-2.5051	0.0209	-0.0136
4.2491	0.0552	-2.8083	0.1509
0	1.0000	0	0

Closed-loop Eigenvalue Matrix

-1.50 + 1.50i	0	0	0
0	-1.50 - 1.50i	0	0
0	0	-2.50	0
0	0	0	-0.0050

Closed-loop Eigenvector Matrix

1.0000	1.0000	-0.0000	-0.0000
-0.0225 + 0.0000i	-0.0225 - 0.0000i	1.0000	-0.0050
1.4036 - 1.6084i	1.4036 + 1.6084i	-0.0168	0.0537
0.0075 + 0.0075i	0.0075 - 0.0075i	-0.4000	1.0000

Figure 2 - DEA Design Closed-Loop System Matrices and Eigenspace.

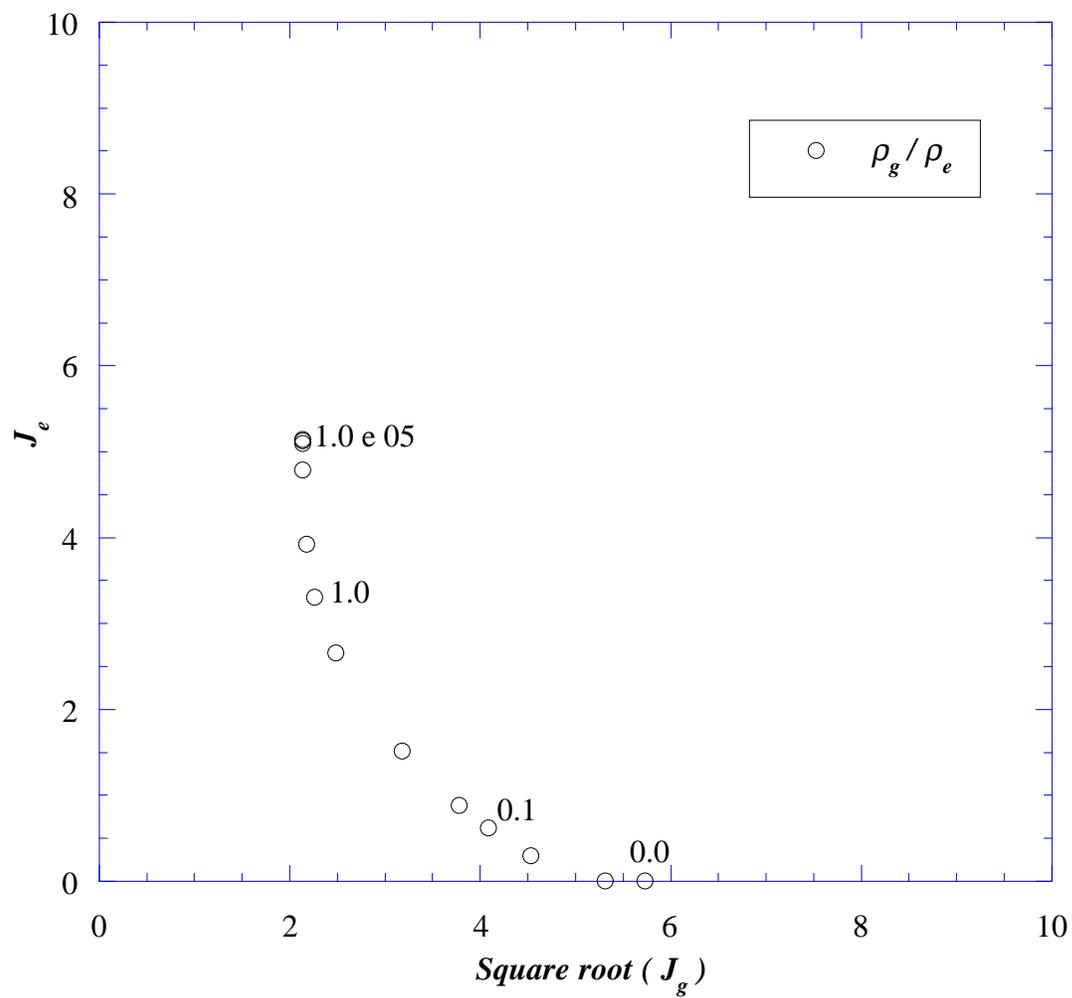


Figure 3 - Eigenvector Error (J_e) Versus RMS Gain Magnitude (Square root (J_g)).

Gain Matrix

G =

-0.3096	-0.8200	-5.1642	-0.6697
0.0230	0.1138	-0.6396	2.1203

gainmag =

5.73

Closed-Loop System Matrix

$A+B*inv(I_m-G*N)*G*M$

-0.1916	0.0044	-0.9324	0.0501
-0.0519	-2.5051	0.0209	-0.0136
4.2491	0.0552	-2.8083	0.1509
0	1.0000	0	0

Closed-Loop Eigenvalue Matrix

-1.5000 + 1.5000i	0	0	0
0	-1.5000 - 1.5000i	0	0
0	0	-2.5000	0
0	0	0	-0.0050

Closed-Loop Eigenvector Matrix

1.0000	1.0000	-0.0000	-0.0000
-0.0225 + 0.0000i	-0.0225 - 0.0000i	1.0000	-0.0050
1.4036 - 1.6084i	1.4036 + 1.6084i	-0.0168	0.0537
0.0075 + 0.0075i	0.0075 - 0.0075i	-0.4000	1.0000

Figure 4 - GWEA Design $\rho_g / \rho_e = 0.0$.

Gain Matrix

G =

-0.5988	0.1195	-3.6945	-0.2224
0.3561	-0.9005	-0.8321	0.9629

gainmag =

4.08

Closed-Loop System Matrix

A+B*inv(Im-G*N)*G*M

-0.1927	-0.0053	-0.9384	0.0520
-4.2145	-2.6323	1.0924	-0.0081
4.7782	0.4762	-2.6800	0.0720
0	1.0000	0	0

Closed-Loop Eigenvalue Matrix

-1.5000 + 1.5000i	0	0	0
0	-1.5000 - 1.5000i	0	0
0	0	-2.5000	0
0	0	0	-0.0050

Closed-Loop Eigenvector Matrix

1.0000	1.0000	-0.1044	0.0148
-1.5835 + 0.5681i	-1.5835 - 0.5681i	1.0000	-0.0050
1.4418 - 1.5829i	1.4418 + 1.5829i	-0.2844	0.0525
0.7172 + 0.3384i	0.7172 - 0.3384i	-0.4000	1.0000

Figure 5 - GWEA Design $\rho_g/\rho_e = 0.1$.

Gain Matrix

G =

-1.0725	0.4415	-0.8696	-0.0573
0.3095	-0.5105	-1.0925	1.2173

gainmag =
2.25

Closed-Loop System Matrix

A+B*inv(Im-G*N)*G*M

-0.1778	-0.0017	-0.9450	0.0515
-10.0658	-2.8715	1.5224	-0.0120
4.8368	0.3527	-2.4556	0.0918
0	1.0000	0	0

Closed-Loop Eigenvalue Matrix

-1.5000 + 1.5000i	0	0	0
0	-1.5000 - 1.5000i	0	0
0	0	-2.5000	0
0	0	0	-0.0050

Closed-Loop Eigenvector Matrix

1.0000	1.0000	-0.0637	0.0083
-3.4556 + 2.0378i	-3.4556 - 2.0378i	1.0000	-0.0050
1.5052 - 1.5652i	1.5052 + 1.5652i	-0.1801	0.0530
1.8311 + 0.4726i	1.8311 - 0.4726i	-0.4000	1.0000

Figure 6 - GWEA Design $\rho_g / \rho_e = 1.0$.

Gain Matrix

G =

-1.0238	0.3554	-0.2791	0.1474
0.1539	-0.3809	-0.9694	1.4704

gainmag =
2.13

Closed-Loop System Matrix
A+B*inv(Im-G*N)*G*M

-0.1662	0.0023	-0.9402	0.0511
-11.0128	-2.7160	1.2526	-0.0054
4.4611	0.1667	-2.6228	0.1097
0	1.0000	0	0

Closed-Loop Eigenvalue Matrix

-1.5000 + 1.5000i	0	0	0
0	-1.5000 - 1.5000i	0	0
0	0	-2.5000	0
0	0	0	-0.0050

Closed-Loop Eigenvector Matrix

1.0000	1.0000	-0.0301	0.0068
-3.7635 + 3.0195i	-3.7635 - 3.0195i	1.0000	-0.0050
1.5322 - 1.5744i	1.5322 + 1.5744i	-0.0940	0.0532
2.2610 + 0.2480i	2.2610 - 0.2480i	-0.4000	1.0000

Figure 7 - GWEA Design $\rho_g/\rho_e = 1.0e05$.

Desired Eigenvalues

$$\lambda_1 = -0.005 \quad (\text{spiral})$$

$$\lambda_2 = -2.5 \quad (\text{roll})$$

$$\lambda_{3,4} = -1.5 \pm 1.5j \quad (\omega_n = 2.1 \text{ (rad/sec)}, \zeta = 0.7) \quad (\text{dutch roll})$$

Desired Eigenvectors

Mode 1	Mode 2	Mode 3	Mode 4
(spiral)	(roll)	(dutch roll)	(dutch roll)
0	0	1.0	1.0
x	1	x	x
x	x	x	x
1	x	$0.0075 + 0.0075j$	$0.0075 - 0.0075j$

(x denotes eigenvector elements that are not weighted in the cost function)

Table 1 - Desired Eigenvalues and Eigenvectors.

Mode 1	Mode 2	Mode 3	Mode 4
(spiral)	(roll)	(dutch roll)	(dutch roll)
0	0	1	1
0	1	0	0
0	0	0	0
1	0	0	0

Table 2 - Reference Weighting of Desired Eigenvector Elements.

ρ_e	ρ_g	ρ_g / ρ_e	Sqrt(J_g)	J_e
1.00	0.0	0.0	5.7274	1.99e-17
1.00	0.010	0.01000	5.3075	0.00767
1.00	0.0667	0.0667	4.5356	0.2926
1.00	0.10	0.100	4.0780	0.6168
1.00	0.125	0.125	3.7757	0.8785
1.00	0.20	0.200	3.1843	1.5105
1.00	0.50	0.500	2.4804	2.6575
1.00	1.00	1.00	2.2663	3.3091
1.00	2.00	2.00	2.1821	3.9256
1.00	10.0	10.0	2.1352	4.7903
0.010	1.00	100.0	2.1319	5.0980
0.0010	1.00	1000.0	2.1318	5.1330
1.00e-05	1.00	1.00e05	2.1318	5.1372

Table 3 - GWEA Design Points.