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Closed-Form Evaluation of Mutual Coupling in a Planar Array of Circular Apertures

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Abstract

The integral expression for the mutual admittance between circular apertures in a planar array is evaluated in closed form. Very good accuracy is realized when compared with values that were obtained by numerical integration. Utilization of this closed-form expression, for all element pairs that are separated by more than one element spacing, yields extremely accurate results and significantly reduces the computation time that is required to analyze the performance of a large electronically scanning antenna array.

Introduction

The wide flexibility available in the design of antenna arrays is very useful in applications where factors such as beam shaping, side-lobe control, and rapid beam steering are of prime consideration; however, the implementation of a good design can become quite complicated as a result of the effects of mutual interaction between closely spaced radiating elements. These interactions are evident as (1) a distortion of the radiation pattern, (2) an element-driving impedance that varies as the array is phased to point the beam in different directions, and (3) a polarization variation with scan angle in an array with elements that can support more than one sense of polarization. The degree to which the interelement coupling affects the performance of the array will depend upon the element type, the polarization and excitation of each element, the geometry of the array, and the surrounding environment. To accurately model the effects of mutual interelement coupling in the design of a phased-array antenna, the analysis must include all these factors.

Since interelement coupling in phased arrays is a near-field phenomenon, an accurate analytical model is generally formulated such that the resulting expression involves either a single or a double integration in the spectral domain. This integral formulation can readily be evaluated numerically with the aid of high-speed computers; however, the computation time can still become prohibitively large. This substantially increased computation time for large arrays is primarily a result of the need to calculate the mutual coupling between all possible pair combinations of the array and the highly oscillatory nature of the integrand to be evaluated, which oscillates more rapidly and converges more slowly as the separation between element pairs increases.

The focus of this paper is to illustrate a technique for developing an accurate closed-form evaluation of the integral formulation for mutual coupling between circular-aperture elements in a planar array. In particular, the final results in this paper are limited to identical circular elements whose aperture fields are restricted to that of the dominant mode of a circular waveguide of the same cross section; however, the approach is applicable to other aperture fields that can be represented in Bessel-function form.

Symbols

$A(k_x,k_y,z)$	solution to wave equation in spectral domain
a	radius of circular aperture
a_i	complex amplitude of modal field incident on ith aperture
$a_1, a_2, a_3, ..., a_N$	complex amplitude of modal field incident on aperture (1, 2, 3, ..., N)
b_i	complex amplitude of modal field reflected from ith aperture
$b_1, b_2, b_3, ..., b_N$	complex amplitude of modal field reflected from aperture (1, 2, 3, ..., N)
D	diameter of circular aperture
$\mathbf{E}^{(i)}(x,y,z)$	vector electric field due to excitation of ith aperture

$\mathbf{E}^{(j)}(x,y,z)$	vector electric field due to excitation of jth aperture
$\mathbf{E}^{(i)}(k_x,k_y,z)$	bidimensional Fourier transform of $\mathbf{E}^{(i)}(x,y,z)$
$\mathbf{E}^{(j)}(k_x,k_y,z)$	bidimensional Fourier transform of $\mathbf{E}^{(j)}(x,y,z)$
E_x,E_y	x and y scalar components of vector electric field $\mathbf{E}$
e	base of natural logarithms (≈2.718281828459)
$F(k_x,k_y,z)$	solution to wave equation in spectral domain
$f(\beta)$	function defined in equation (22)
$g(\beta)$	function defined in equation (14)
$\mathbf{H}^{(i)}(x,y,z)$	vector magnetic field due to excitation of ith aperture
$\mathbf{H}^{(j)}(x,y,z)$	vector magnetic field due to excitation of jth aperture
$\mathbf{H}^{(i)}(k_x,k_y,z)$	bidimensional Fourier transform of $\mathbf{H}^{(i)}(x,y,z)$
$\mathbf{H}^{(j)}(k_x,k_y,z)$	bidimensional Fourier transform of $\mathbf{H}^{(j)}(x,y,z)$
H_x,H_y	x and y scalar components of vector magnetic field $\mathbf{H}$
$J_\nu(\ )$	Bessel function of first kind and of order ν
$J'_\nu(\ )$	first derivative of $J_\nu(\ )$ with respect to the argument
j	$= \sqrt{-1}$
k_x	Fourier transform variable in x-direction
k_y	Fourier transform variable in y-direction
k_0	wave propagation constant in free space, $2\pi/\lambda_0$
m	index for products as defined in equation (12)
N	total number of elements in array
n	index for summation as defined in equation (12)
R	radial distance between aperture centers in cylindrical coordinates
r	radial distance in spherical coordinates
S_i	area of ith aperture
S_{ij}	coefficients of scattering matrix
$T_\nu(k_0\beta R)$	quantity defined in equation (12)
V_i	complex amplitude of modal voltage excitation of ith aperture
V_j	complex amplitude of modal voltage excitation of jth aperture
$W_1(\beta)$	quantity defined in equation (17)
$W_2(\beta)$	quantity defined in equation (18)
x,y,z	spatial variables in Cartesian coordinates
x_i,y_i	Cartesian coordinates of ith aperture center
x'_{11}	first zero of derivative of $J_1(x)$, ≈1.84118
Y_i	modal characteristic admittance for ith aperture
Y_{ij}	mutual admittance between ith and jth apertures in an array
Y_{12}	mutual admittance between apertures 1 and 2
y_{ij}	coefficients of normalized admittance matrix, Y_{ij}/Y_i
$\mathbf{z}$	unit vector in z-direction

α,β	variables of integration
Γ_i	complex reflection coefficient of *i*th element in array, b_i/a_i
δ_{ij}	coefficients of unit matrix
ε	permittivity of medium outside of aperture plane
ε_r	$= \varepsilon/\varepsilon_0$
ε_0	permittivity of free space
$\zeta(\beta)$	quantity defined in equation (20)
θ	angle with respect to *z*-axis in spherical coordinates
θ_0,ϕ_0	antenna-beam pointing direction in spherical coordinates
λ_0	wavelength of electromagnetic field in free space
μ	permeability of medium outside of aperture plane
μ_0	permeability of free space
μ_r	$= \mu/\mu_0$
ν	order of Bessel function
$\xi(\beta)$	quantity defined in equation (19)
ξ_0,ζ_0,σ_0	quantities defined in equation (33)
π	ratio of circumference of a circle to its diameter
ϕ	angular position of center of aperture 2 with respect to center of aperture 1
ϕ_p	angular polarization of aperture 2 with respect to aperture 1
ψ_{n0},ψ_{n2}	quantities defined in equation (34)
ω	electromagnetic wave frequency, rad/sec

Analysis

The general analytical formulation for the interelement mutual coupling in planar arrays of arbitrary apertures has been developed. (See ref. 1.) The effects of mutual coupling are determined by computing the self and mutual admittances among all the elements of the array to form a complex admittance matrix. This admittance matrix is then operated on to determine the complex scattering matrix for the array. The scattering matrix represents the relationship between the amplitudes and phases of all the feed-waveguide modal fields that are incident on and reflected from the radiating apertures. This complex scattering matrix allows one to completely characterize the performance of the antenna for any amplitude and phase excitation.

The mutual admittance between the *i*th and *j*th apertures of the array can be determined from the reaction between the electric field of the *i*th aperture and the magnetic field of the *j*th aperture. In general,

$$Y_{ij} = \frac{1}{V_i V_j}\iint_{S_i}\left[\mathbf{E}^{(i)}(x, y, 0) \times \mathbf{H}^{(j)}(x, y, 0)\right] \cdot \mathbf{z}\ dx\ dy \tag{1}$$

where $\mathbf{E}^{(i)}(x,y,0)$ is the vector electric field in the *i*th aperture with all others long circuited, and $\mathbf{H}^{(j)}(x,y,0)$ is the vector magnetic field that would exist at the *i*th aperture with all apertures short circuited except the *j*th. The fields are evaluated at the aperture plane ($z = 0$). However, for purposes of this analysis (which will become obvious), equation (1) is rewritten as

$$Y_{ij} = \frac{1}{V_i V_j} \iint_{S_i} \left[\mathbf{E}^{(i)}(x, y, 0) \times \lim_{z \to 0} \mathbf{H}^{(j)}(x, y, z) \right] \cdot \mathbf{z} \; dx \; dy \tag{2}$$

In the Fourier spectral domain, equation (2) can be expressed in an equivalent form (ref. 1) as

$$Y_{ij} = \frac{1}{4\pi^2 V_i V_j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\mathbf{E}^{(i)}(k_x, k_y, 0) \times \lim_{z \to 0} \mathbf{H}^{(j)}(-k_x, -k_y, z) \right] \cdot \mathbf{z} \; dk_x \; dk_y \tag{3}$$

By performing the vector multiplications and by indicating that the integrations are to be performed before taking the limit, the admittance expression is rewritten as

$$Y_{ij} = \frac{1}{4\pi^2 V_i V_j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[E_x^{(i)}(k_x, k_y, 0) H_y^{(j)}(-k_x, -k_y, z) - E_y^{(i)}(k_x, k_y, 0) H_x^{(j)}(-k_x, -k_y, z) \right] dk_x \; dk_y \tag{4}$$

In the spectral domain, the transverse components of the transformed electric and magnetic fields are related to the transformed solutions to the wave equations as

$$\left.\begin{aligned} E_x(k_x, k_y, z) &= \frac{-k_x}{\omega\varepsilon(z)} A'(k_x, k_y, z) + jk_y F(k_x, k_y, z) \\ E_y(k_x, k_y, z) &= \frac{-k_y}{\omega\varepsilon(z)} A'(k_x, k_y, z) - jk_x F(k_x, k_y, z) \\ H_x(k_x, k_y, z) &= \frac{-k_x}{\omega\mu(z)} F'(k_x, k_y, z) - jk_y A(k_x, k_y, z) \\ H_y(k_x, k_y, z) &= \frac{-k_y}{\omega\mu(z)} F'(k_x, k_y, z) + jk_x A(k_x, k_y, z) \end{aligned}\right\} \tag{5}$$

where the primes on A and F denote differentiation with respect to z. The admittance expression can then be rewritten as

$$Y_{ij} = \frac{j}{4\pi^2 V_i V_j} \lim_{z \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_x^2 + k_y^2) \left[\frac{A'(k_x, k_y, 0) A(-k_x, -k_y, z)}{\omega\varepsilon(0)} + \frac{F(k_x, k_y, 0) F'(-k_x, -k_y, z)}{\omega\mu(z)} \right] dk_x \; dk_y \tag{6}$$

When the first term is multiplied and divided by $A'(-k_x,-k_y,0)$, and the second term is multiplied and divided by $F(-k_x,-k_y,0)$, equation (6) becomes

$$\begin{aligned} Y_{ij} = \frac{j}{4\pi^2 V_i V_j} \lim_{z \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_x^2 + k_y^2) \Bigg\{ & A'(k_x, k_y, 0) A'(-k_x, -k_y, 0) \left[\frac{A(-k_x, -k_y, z)}{\omega\varepsilon(0) A'(-k_x, -k_y, 0)} \right] \\ & + F(k_x, k_y, 0) F(-k_x, -k_y, 0) \left[\frac{F'(-k_x, -k_y, z)}{\omega\mu(z) F(-k_x, -k_y, 0)} \right] \Bigg\} \; dk_x \; dk_y \end{aligned} \tag{7}$$

Solving equation (5) independently for the ith and jth apertures and substituting into equation (7) yields the following general expression for the mutual admittance between two apertures whose electric-field distributions are known:

$$
\begin{aligned}
Y_{ij} = {} & \frac{k_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}}{4\pi^2 V_i V_j} \lim_{z \to 0} \int_{\alpha=0}^{2\pi}\int_{\beta=0}^{\infty}\left\{\left[\frac{k_0\varepsilon_r(0)A(\alpha,\beta,z)}{jA'(\alpha,\beta,0)}\right][E_x^{(i)}(\alpha,\beta,0)\cos\alpha + E_y^{(i)}(\alpha,\beta,0)\sin\alpha]\right. \\
& \times [E_x^{(j)}(\alpha,-\beta,0)\cos\alpha + E_y^{(j)}(\alpha,-\beta,0)\sin\alpha] + \left[\frac{jF'(\alpha,\beta,z)}{k_0\mu_r(z)F(\alpha,\beta,0)}\right] \\
& \left.\times [E_x^{(i)}(\alpha,\beta,0)\sin\alpha - E_y^{(i)}(\alpha,\beta,0)\cos\alpha][E_x^{(j)}(\alpha,-\beta,0)\sin\alpha - E_y^{(j)}(\alpha,-\beta,0)\cos\alpha]\right\}\beta\, d\beta\, d\alpha
\end{aligned} \tag{8}
$$

where a change of variables has been made such that $k_x = k_0\beta \cos\alpha$ and $k_y = k_0\beta \sin\alpha$. The solutions to the free-space wave equations in the Fourier transform domain yield

$$
\left.
\begin{aligned}
\left[\frac{k_0\varepsilon_r(0)A(\alpha,\beta,z)}{jA'(\alpha,\beta,0)}\right] &= \frac{\exp\left(-jk_0 z\sqrt{1-\beta^2}\right)}{\sqrt{1-\beta^2}} \\
\left[\frac{jF'(\alpha,\beta,z)}{k_0\mu_r(z)F(\alpha,\beta,0)}\right] &= \sqrt{1-\beta^2}\exp\left(-jk_0 z\sqrt{1-\beta^2}\right)
\end{aligned}
\right\} \tag{9}
$$

For circular apertures, whose field distributions are those of the circular waveguide modes, the integration on α in equation (8) can be readily evaluated in terms of Bessel functions. The mutual admittance expression for circular apertures then reduces to a single integration on β.

To demonstrate the method of evaluating the closed-form expression for mutual admittance, the remainder of this report is limited to identical circular apertures with only the dominant transverse electric mode (TE_{11}) aperture fields. The mathematical development for other circular waveguide modal fields and unequal size apertures would proceed in the same manner. Assuming identical TE_{11} mode circular apertures, equation (8) can be written as

$$
\begin{aligned}
Y_{ij} = {} & \frac{2\sqrt{\frac{\varepsilon_0}{\mu_0}}}{x_{11}'^2 - 1}\lim_{z\to 0}\int_0^{\infty}\exp\left(-jk_0 z\sqrt{1-\beta^2}\right)\left\{\sqrt{1-\beta^2}\left[\frac{x_{11}'^2 k_0 a J_1'(k_0 a\beta)}{x_{11}'^2 - k_0^2 a^2\beta^2}\right]^2\right. \\
& \times [J_0(k_0\beta R)\cos\phi_p - J_2(k_0\beta R)\cos(2\phi - \phi_p)] \\
& \left. + \frac{J_1^2(k_0 a\beta)}{\beta^2\sqrt{1-\beta^2}}[J_0(k_0\beta R)\cos\phi_p + J_2(k_0\beta R)\cos(2\phi-\phi_p)]\right\}\beta\, d\beta
\end{aligned} \tag{10}
$$

Although the two apertures are identical in size and excitation, they may be polarized differently with respect to each other (as denoted by the relative polarization angle ϕ_p). The geometry for the two circular apertures is illustrated in figure 1.

To evaluate equation (10) in closed form, the semiconvergent series of Hankel (ref. 2, pp. 137 and 138) is first utilized to express the Bessel functions (with arguments $k_0\beta R$) in a series form as

$$
J_\nu(k_0\beta R) = \frac{\exp\left(-j\frac{\pi}{4}\right)}{j^\nu\sqrt{2\pi}\sqrt{k_0\beta R}}[T_\nu(-k_0\beta R)\exp(jk_0\beta R) - jT_\nu(k_0\beta R)\exp(-jk_0\beta R)] \tag{11}
$$

where

$$T_{\nu}(k_0\beta R) = 1 + \sum_{n=1}^{\infty}\left\{\frac{(-j)^n}{n!(8k_0\beta R)^n}\prod_{m=1}^{n}[4\nu^2-(2m-1)^2]\right\} \tag{12}$$

and where ν is the order of the Bessel function. (For the special case of TE_{11} mode, ν is either 0 or 2.) When equation (11) is substituted into equation (10) and β is substituted for −β in terms containing $T\nu(-k_0\beta R)$, the integration on β in equation (10) can be extended over the limits of (−∞ to ∞) as

$$Y_{12} = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\varepsilon_0}{\mu_0}}\exp\left(j\frac{\pi}{4}\right)\lim_{z\to 0}\left\{\frac{1}{\sqrt{k_0R}}\int_{-\infty}^{\infty}g(\beta)\exp\left[-jk_0(\beta R+z\sqrt{1-\beta^2})\right]d\beta\right\} \tag{13}$$

where the following additional quantities have been introduced for convenience:

$$g(\beta) = \sqrt{\beta}[G_0(\beta)T_0(k_0\beta R)-G_2(\beta)T_2(k_0\beta R)] \tag{14}$$

where

$$G_0(\beta) = [W_1(\beta)\xi^2(\beta)+W_2(\beta)\zeta^2(\beta)]\cos\phi_p \tag{15}$$

$$G_2(\beta) = [W_1(\beta)\xi^2(\beta)-W_2(\beta)\zeta^2(\beta)]\cos(2\phi-\phi_p) \tag{16}$$

$$W_1(\beta) = \frac{1}{\sqrt{1-\beta^2}} \tag{17}$$

$$W_2(\beta) = \sqrt{1-\beta^2} \tag{18}$$

$$\xi(\beta) = \frac{J_1(k_0a\beta)}{\beta\sqrt{x_{11}^{\prime 2}-1}} \tag{19}$$

$$\zeta(\beta) = \frac{x_{11}^{\prime 2}k_0a\left[J_0(k_0a\beta)-\dfrac{J_1(k_0a\beta)}{k_0a\beta}\right]}{\sqrt{x_{11}^{\prime 2}-1}\left[x_{11}^{\prime 2}-(k_0a\beta)^2\right]} \tag{20}$$

Expressing z and R in spherical coordinates ($z = r\cos\theta$ and $R = r\sin\theta$), equation (13) can be rewritten as

$$Y_{12} = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\frac{1}{k_0r}}\exp\left(j\frac{\pi}{4}\right)\lim_{\theta\to\frac{\pi}{2}}\left\{\frac{1}{\sqrt{\sin\theta}}\int_{-\infty}^{\infty}g(\beta)\exp[jrf(\beta)]\,d\beta\right\} \tag{21}$$

where

$$f(\beta) = -k_0\left(\beta\sin\theta+\sqrt{1-\beta^2}\cos\theta\right) \tag{22}$$

The integral in equation (21) is now in a form that can be readily evaluated, for large values of r, by the saddle-point method (ref. 3, pp. 305 to 307) as

$$\int_{-\infty}^{\infty} g(\beta)\exp[jrf(\beta)]\,d\beta \approx g(\beta_0)\exp[jrf(\beta_0)]\sqrt{\frac{2\pi}{-jrf''(\beta_0)}}\left\{1+\frac{1}{2jrf''(\beta_0)}\right.$$
$$\left.\times\left[\frac{f'''(\beta_0)g'(\beta_0)}{f''(\beta_0)g(\beta_0)}+\frac{f''''(\beta_0)}{4f''(\beta_0)}-\frac{5}{12}\left(\frac{f'''(\beta_0)}{f''(\beta_0)}\right)^2-\frac{g''(\beta_0)}{g(\beta_0)}+\ldots\right]\right\} \qquad (23)$$

The primes on f and g denote differentiation with respect to β, and β_0 is the saddle point as determined from

$$\left.\begin{aligned} f'(\beta_0) &= 0 \\ f''(\beta_0) &\neq 0 \end{aligned}\right\} \qquad (24)$$

Therefore,

$$\beta_0 = \sin\theta \qquad (25)$$

The modification in equation (2) allowed the saddle point to be determined as defined in equation (24); thus, the integral evaluation can be performed.

The evaluation of the integral in equation (23) requires taking partial derivatives (up to the fourth order) with respect to the integration variable β and evaluating these derivatives at the saddle point β_0. The quantities of interest are

$$\left.\begin{aligned} f(\beta_0) &= -k_0 \\ f'(\beta_0) &= 0 \\ f''(\beta_0) &= \frac{k_0}{\cos^2\theta} \\ f'''(\beta_0) &= \frac{3k_0\sin\theta}{\cos^4\theta} \\ f''''(\beta_0) &= \frac{3k_0(4\sin^2\theta+1)}{\cos^6\theta} \end{aligned}\right\} \qquad (26)$$

$$\left.\begin{aligned} W_1(\beta_0) &= \frac{1}{\cos\theta} \\ W_1'(\beta_0) &= \frac{\sin\theta}{\cos^3\theta} \\ W_1''(\beta_0) &= \frac{2\sin^2\theta+1}{\cos^5\theta} \end{aligned}\right\} \qquad (27)$$

and

$$\left.\begin{aligned} W_2(\beta_0) &= \cos\theta \\ W_2'(\beta_0) &= \frac{-\sin\theta}{\cos\theta} \\ W_2''(\beta_0) &= \frac{-1}{\cos^3\theta} \end{aligned}\right\} \tag{28}$$

The primes on W_1 and W_2 denote differentiation with respect to β. Taking derivatives of g with respect to β, setting $\beta = \beta_0$, and dropping the functional notations yields

$$g(\beta_0) = \sqrt{\beta_0}\left[T_0(W_1\xi^2 + W_2\zeta^2)\cos\phi_p - T_2(W_1\xi^2 - W_2\zeta^2)\cos(2\phi - \phi_p)\right] \tag{29}$$

$$\begin{aligned} g'(\beta_0) = {} & \frac{1}{2\sqrt{\beta_0}}\left[T_0(W_1\xi^2 + W_2\zeta^2)\cos\phi_p - T_2(W_1\xi^2 - W_2\zeta^2)\cos(2\phi - \phi_p)\right] \\ & + \sqrt{\beta_0}\Big\{[T_0'(W_1\xi^2 + W_2\zeta^2) + T_0(W_1'\xi^2 + 2W_1\xi\xi' + W_2'\zeta^2 + 2W_2\zeta\zeta')]\cos\phi_p \\ & - [T_2'(W_1\xi^2 - W_2\zeta^2) + T_2(W_1'\xi^2 + 2W_1\xi\xi' - W_2'\zeta^2 - 2W_2\zeta\zeta')]\cos(2\phi - \phi_p)\Big\} \end{aligned} \tag{30}$$

and

$$\begin{aligned} g''(\beta_0) = {} & \frac{-1}{4\beta_0\sqrt{\beta_0}}[T_0(W_1\xi^2 + W_2\zeta^2)\cos\phi_p - T_2(W_1\xi^2 - W_2\zeta^2)\cos(2\phi - \phi_p)] \\ & + \frac{1}{\sqrt{\beta_0}}\{[T_0'(W_1\xi^2 + W_2\zeta^2) + T_0(W_1'\xi^2 + 2W_1\xi\xi' + W_2'\zeta^2 + 2W_2\zeta\zeta')]\cos\phi_p \\ & - [T_2'(W_1\xi^2 - W_2\zeta^2) + T_2(W_1'\xi^2 + 2W_1\xi\xi' - W_2'\zeta^2 - 2W_2\zeta\zeta')]\cos(2\phi - \phi_p)\} \\ & + \sqrt{\beta_0}\{[T_0''(W_1\xi^2 + W_2\zeta^2) + 2T_0'(W_1'\xi^2 + 2W_1\xi\xi' + W_2'\zeta^2 + 2W_2\zeta\zeta') \\ & + T_0(W_1''\xi^2 + 4W_1'\xi\xi' + 2W_1\xi'\xi' + 2W_1\xi\xi'' + W_2''\zeta^2 + 4W_2'\zeta\zeta' + 2W_2\zeta'\zeta' + 2W_2\zeta\zeta'')]\cos\phi_p \\ & - [T_2''(W_1\xi^2 - W_2\zeta^2) + 2T_2'(W_1'\xi^2 + 2W_1\xi\xi' - W_2'\zeta^2 - 2W_2\zeta\zeta') \\ & + T_2(W_1''\xi^2 + 4W_1'\xi\xi' - 2W_1\xi'\xi' - 2W_1\xi\xi'' - W_2''\zeta^2 - 4W_2'\zeta\zeta' - 2W_2\zeta'\zeta' - 2W_2\zeta\zeta'')]\cos(\phi - \phi_p)\} \end{aligned} \tag{31}$$

All primes in equations (30) and (31) denote derivatives with respect to β evaluated at $\beta = \beta_0$. Substituting equations (26) to (31) into equation (23) and evaluating equation (21) at $\theta = \pi/2$ yields (after considerable algebraic manipulation)

$$
\begin{aligned}
Y_{12} = {} & 2j\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{1}{x_{11}'^2-1}\left[\frac{\exp(-jk_0R)}{k_0R}\right] \\
& \times\left(\xi_0^2\left\{[\cos\phi_p-\cos(2\phi-\phi_p)]+\sum_{n=1}^{\infty}[\psi_{n0}\cos\phi_p-\psi_{n2}\cos(2\phi-\phi_p)]\right\}\right) \\
& -2j\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{1}{x_{11}'^2-1}\left[\frac{\exp(-jk_0R)}{k_0R}\right]\left(\frac{j}{8k_0R}\right) \\
& \times\left(\xi_0^2\left\{[\cos\phi_p-\cos(2\phi-\phi_p)]+\sum_{n=1}^{\infty}(n-1)[\psi_{n0}\cos\phi_p-\psi_{n2}\cos(2\phi-\phi_p)]\right\}\right. \\
& -8\zeta_0^2\left\{[\cos\phi_p+\cos(2\phi-\phi_p)]+\sum_{n=1}^{\infty}[\psi_{n0}\cos\phi_p+\psi_{n2}\cos(2\phi-\phi_p)]\right\} \\
& \left.+8\xi_0\sigma_0\left\{[\cos\phi_p-\cos(2\phi-\phi_p)]+\sum_{n=1}^{\infty}[\psi_{n0}\cos\phi_p-\psi_{n2}\cos(2\phi-\phi_p)]\right\}\right)
\end{aligned}
\tag{32}
$$

where

$$
\left.
\begin{aligned}
\xi_0 &= J_1(k_0a) \\
\zeta_0 &= \frac{x_{11}'^2k_0a}{x_{11}'^2-k_0^2a^2}\left[J_0(k_0a)-\frac{J_1(k_0a)}{k_0a}\right] \\
\sigma_0 &= J_0(k_0a)-\frac{(k_0a+1)J_1(k_0a)}{k_0a}
\end{aligned}
\right\}
\tag{33}
$$

and

$$
\left.
\begin{aligned}
\psi_{n0} &= \frac{(-j)^n}{n!(8k_0R)^n}\prod_{m=1}^{n}[-(2m-1)^2] \\
\psi_{n2} &= \frac{(-j)^n}{n!(8k_0R)^n}\prod_{m=1}^{n}[16-(2m-1)^2]
\end{aligned}
\right\}
\tag{34}
$$

Retaining only the terms in $1/R$, $1/R^2$, and $1/R^3$, the mutual-admittance expression becomes

$$
\begin{aligned}
Y_{12} = {} & 2j\exp(-jk_0R)\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{1}{x_{11}'^2-1}\left(\frac{1}{k_0R}\xi_0^2[\cos\phi_p-\cos(2\phi-\phi_p)]\right. \\
& +\frac{j}{k_0^2R^2}\left\{2\xi_0^2\cos(2\phi-\phi_p)+\zeta_0^2[\cos\phi_p+\cos(2\phi-\phi_p)]\right. \\
& \left.-\xi_0\sigma_0[\cos\phi_p-\cos(2\phi-\phi_p)]\right\}+\frac{-1}{128k_0^3R^3}\left\{3\xi_0^2[3\cos\phi_p-35\cos(2\phi-\phi_p)]\right. \\
& \left.\left.+16\zeta_0^2[\cos\phi_p-15\cos(2\phi-\phi_p)]-16\xi_0\sigma_0[\cos\phi_p+15\cos(2\phi-\phi_p)]\right\}\right)
\end{aligned}
\tag{35}
$$

Equation (35) can be used to calculate the complex mutual admittance between a pair of identical circular apertures excited in the dominant TE_{11} mode; however, the self admittance of a single aperture must still be calculated by numerical integration (ref. 1).

The self admittance can be obtained from the integral form of the mutual admittance,

$$Y_{12} = \frac{2\sqrt{\frac{\varepsilon_0}{\mu_0}}}{x_{11}'^2 - 1}\int_0^\infty \left\{ \frac{J_1^2(k_0 a\beta)}{\beta^2\sqrt{1-\beta^2}}\left[J_0(k_0\beta R)\cos\phi_p + J_2(k_0\beta R)\cos(2\phi - \phi_p)\right] \right.$$

$$\left. + \sqrt{1-\beta^2}\left[\frac{x_{11}'^2 k_0 a J_1'(k_0 a\beta)}{x_{11}'^2 - k_0^2 a^2\beta^2}\right]^2 \left[J_0(k_0\beta R)\cos\phi_p - J_2(k_0\beta R)\cos(2\phi - \phi_p)\right] \right\} \beta\, d\beta \tag{36}$$

by making the two apertures coincident (i.e., by setting both the polarization angle and the aperture spacing to zero). The aperture self admittance is then given by

$$Y_{11} = \frac{2\sqrt{\frac{\varepsilon_0}{\mu_0}}}{x_{11}'^2 - 1}\int_0^\infty \left\{ \frac{J_1^2(k_0 a\beta)}{\beta^2\sqrt{1-\beta^2}} + \sqrt{1-\beta^2}\left[\frac{x_{11}'^2 k_0 a J_1'(k_0 a\beta)}{x_{11}'^2 - k_0^2 a^2\beta^2}\right]^2 \right\} \beta\, d\beta \tag{37}$$

In a large array, the self admittance for each element and the mutual admittance for each pair combination is calculated; and the complex scattering matrix for the array can then be determined from the complex normalized admittance matrix by the following matrix relationship:

$$\left[S_{ij}\right] = \left[[\delta_{ij}] - [y_{ij}]\right]\left[[\delta_{ij}] + [y_{ij}]\right]^{-1} \tag{38}$$

with

$$\delta_{ij} = \left\{ \begin{array}{ll} 1 & (i = j) \\ 0 & (i \neq j) \end{array} \right\} \tag{39}$$

and

$$y_{ij} = \frac{Y_{ij}}{Y_i} \tag{40}$$

where S_{ij} is the complex coupling coefficient from aperture j to aperture i, Y_i is the characteristic admittance for the waveguide mode that excites aperture i, and $[\ ]^{-1}$ denotes matrix inversion.

The scattering matrix, in conjunction with the array excitation coefficients, contains the necessary information for describing the performance of a scanning phased-array antenna, including all the interactions between array elements. The scattering matrix gives the relationship between the incident and reflected waveguide modal fields for all the elements of the array as follows:

$$\left.\begin{array}{rcl} b_1 & = & S_{11}a_1 + S_{12}a_2 + S_{13}a_3 + \dots + S_{1N}a_N \\ b_2 & = & S_{21}a_1 + S_{22}a_2 + S_{23}a_3 + \dots + S_{2N}a_N \\ b_3 & = & S_{31}a_1 + S_{32}a_2 + S_{33}a_3 + \dots + S_{3N}a_N \\ \vdots & & \vdots \qquad \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ b_N & = & S_{N1}a_1 + S_{N2}a_2 + S_{N3}a_3 + \dots + S_{NN}a_N \end{array}\right\} \tag{41}$$

The equivalent voltage and current for the *i*th aperture is then expressed as

$$\left.\begin{array}{rcl} V_i & = & a_i + b_i \\ I_i & = & (a_i - b_i)Y_i \end{array}\right\} \tag{42}$$

Therefore, the active admittance of the *i*th element in the array is

$$\frac{I_i}{V_i} = Y_i \frac{a_i - b_i}{a_i + b_i} = Y_i \frac{1 - \Gamma_i}{1 + \Gamma_i} \tag{43}$$

and the active reflection coefficient of the *i*th element is given by

$$\Gamma_i = \frac{b_i}{a_i} = \sum_{j=1}^{N} S_{ij} \frac{a_j}{a_i} \tag{44}$$

Assuming a constant incident power source, the radiated beam from the planar array can be scanned to the angle (θ_0, ϕ_0) by producing a progressive phase shift across the array as follows:

$$a_i = |a_i| \exp[-jk_0(x_i \cos\phi_0 + y_i \sin\phi_0)\sin\theta_0] \tag{45}$$

where x_i and y_i are the Cartesian coordinates of the center of the *i*th aperture. The variation with beam scan for the reflection coefficient of a particular element in the array can be calculated by substituting equation (45) into equation (44) with the scattering coefficients obtained from equation (38).

Results

The closed-form expression (eq. (35)) for mutual admittance was validated by comparing results with those obtained from the numerical integration of equation (36). The scattering-matrix results were used for comparison; in all cases, the self admittance was obtained by numerical integration of equation (37). All computations were performed on an MS-DOS 80486DX2-66 desktop computer using the code CWG (ref. 4); this code was modified to include an option for the closed-form mutual-admittance evaluation.

Two sets of data were used for verification. The first data set was the mutual coupling between two apertures, and the second set was the active reflection coefficient for a large scanning array whose grid geometry is illustrated in figure 2.

The first set of data was obtained for the purpose of establishing the range of applicability by comparing the mutual coupling between two identical apertures for various sizes, orientations, and spacings. These results (plotted in figs. 3 and 4) indicate that very accurate results are consistently obtained for center-to-center spacings greater than the classic far-field distance of $2D^2/\lambda_0$; in many cases, accurate results are obtainable when the apertures are almost touching.

The second set of data was calculated for the active reflection coefficient of the center element in a scanning array. The mutual admittance between all element pairs for a 721-element array (array diameter of 20 wavelengths) was calculated by numerical integration and by the closed-form expression. The calculated reflection coefficient for the center element is plotted against beam scan angle in the two principal planes in figures 5 to 8. The results of these calculations, using numerical integration for all element pairs, were used as the basis for evaluation. When the closed-form expression was used for all element pairs, a small discrepancy was observed, as shown in figures 5 and 6. However, when the admittance matrix was modified to use the numerical integration values for the closest neighbor pairs (i.e., elements spaced $0.714\lambda_0$) and the closed-form expression for all others, the results were extremely accurate, as shown in figures 7 and 8. As a result of the uniform grid geometry of the array, numerical integration of only two mutual-admittance values were required in order to obtain extremely accurate results. Also, as noted in figures 7 and 8, utilization of the closed-form expression resulted in a considerable reduction in computational time required to fill the admittance matrix.

Conclusion

An approach for obtaining a closed-form expression for the mutual admittance between elements in a planar array of apertures was presented. The closed-form expression for circular apertures was developed and compared with results obtained by numerical integration. The judicious use of the closed-form expression, in conjunction with the integral form of the mutual admittance, provides an antenna design and analysis tool that produces extremely accurate results with a significant reduction in computational time for large phased arrays.

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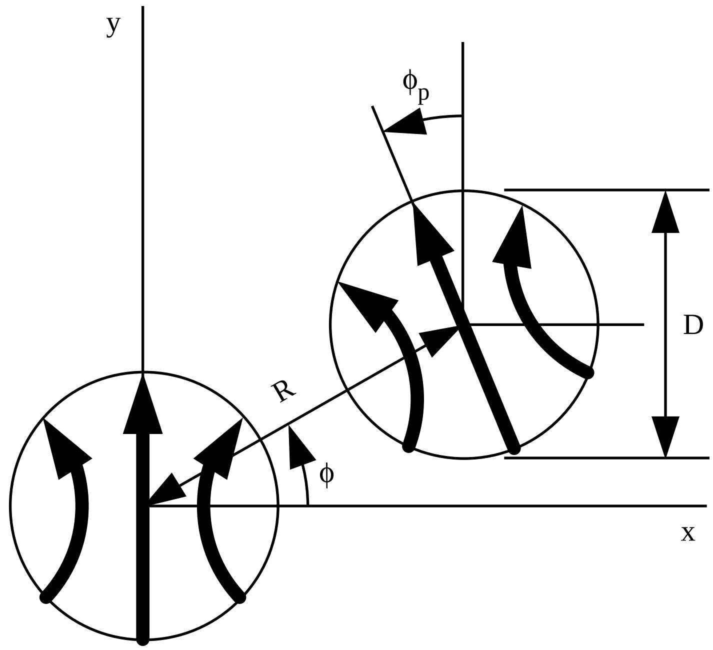

Figure 1. Geometry of dominant transverse electric mode (TE_{11}) excited circular apertures.

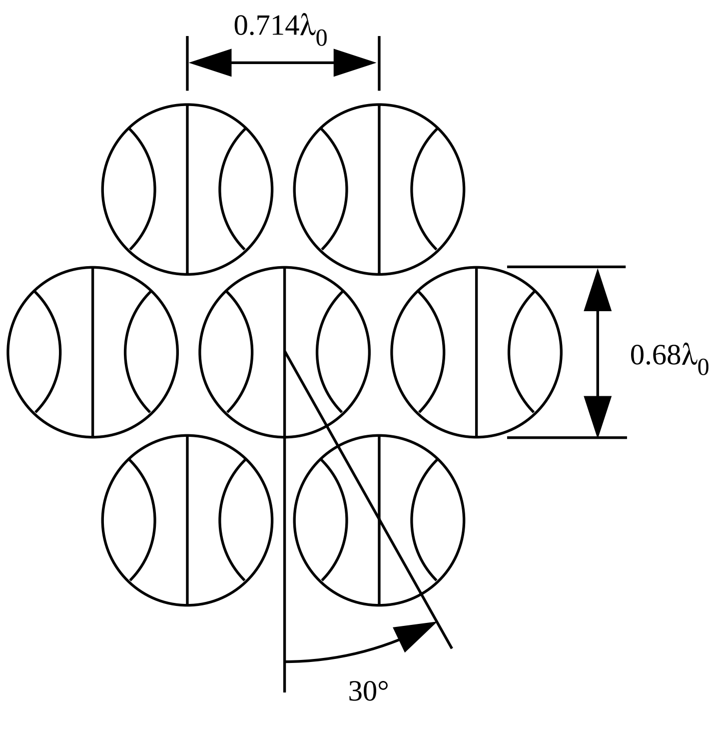

Figure 2. Geometry for equilateral triangular grid array.

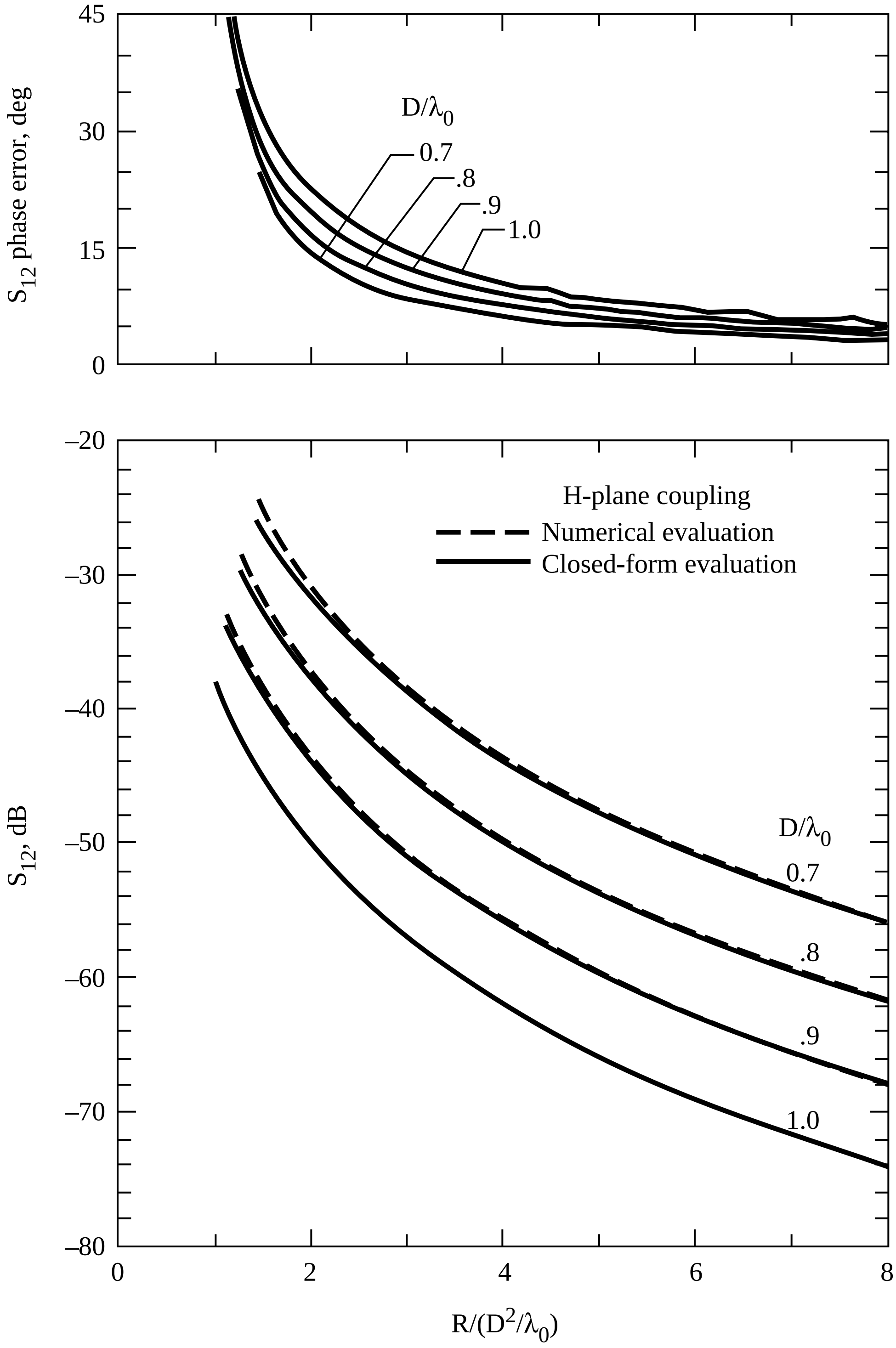

Figure 3. H-plane coupling between circular apertures with TE_{11} mode excitation.

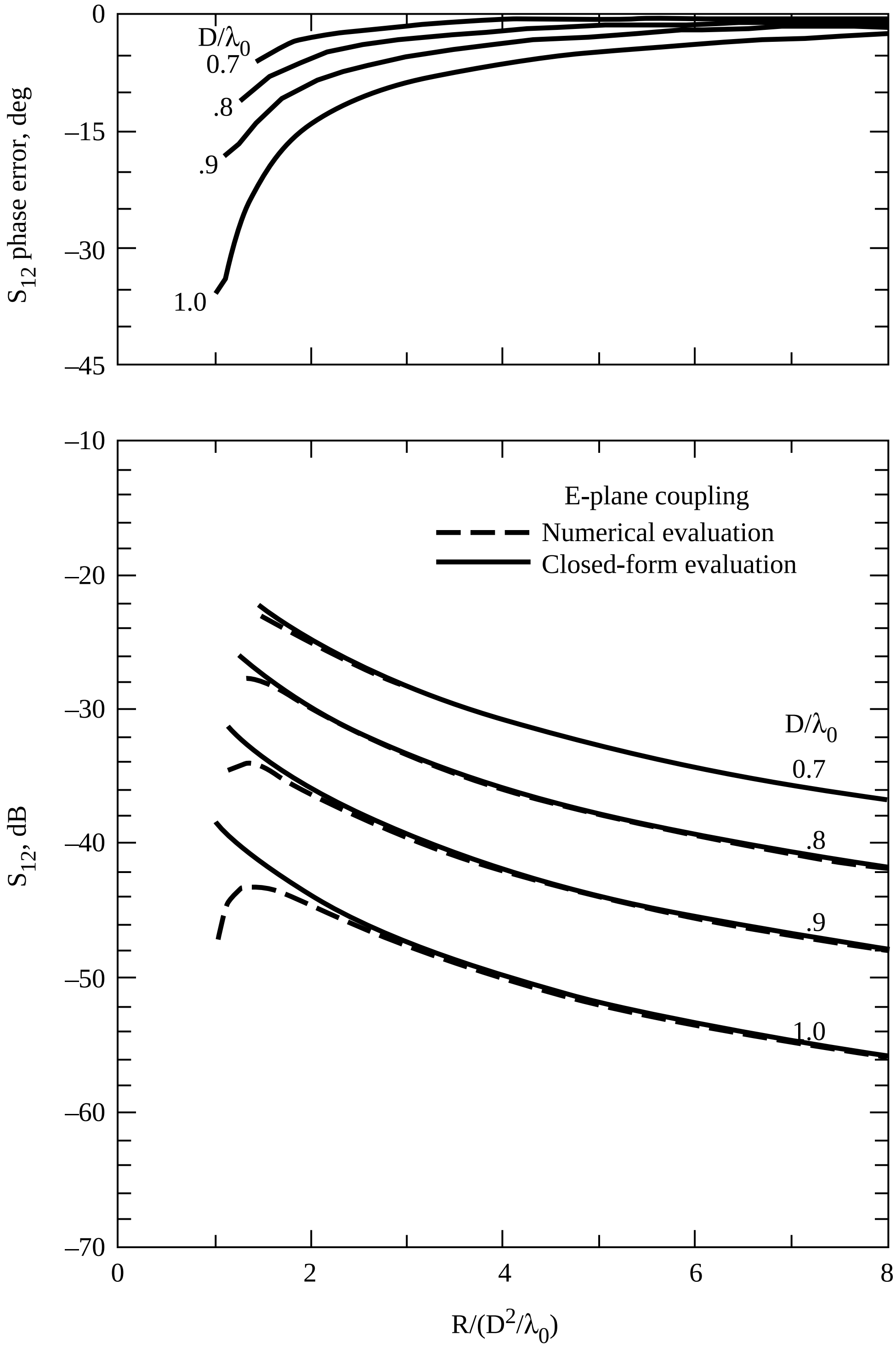

Figure 4. E-plane coupling between circular apertures with TE_{11} mode excitation.

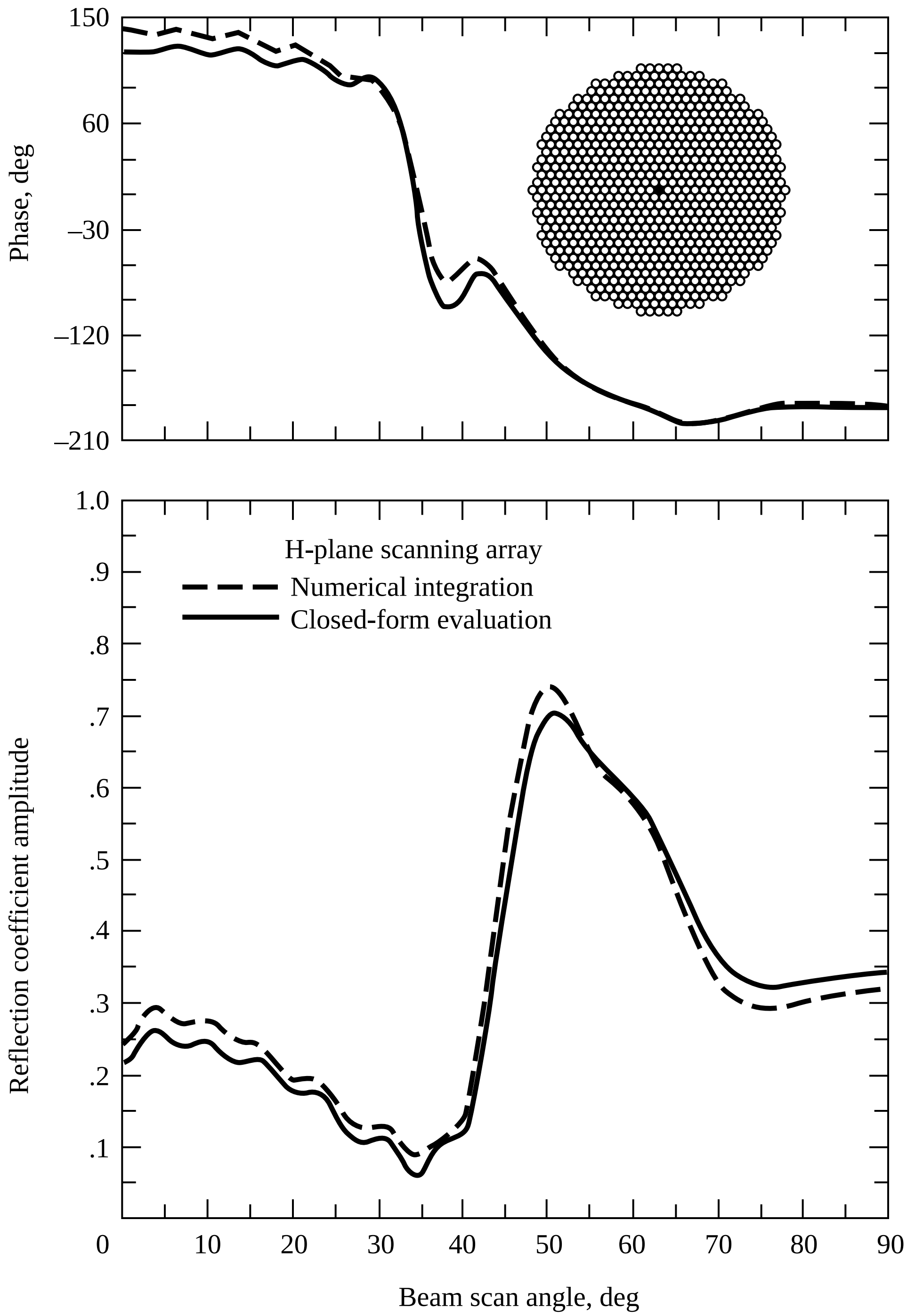

Figure 5. Reflection coefficient versus H-plane scan angle for center element of 721-element array.

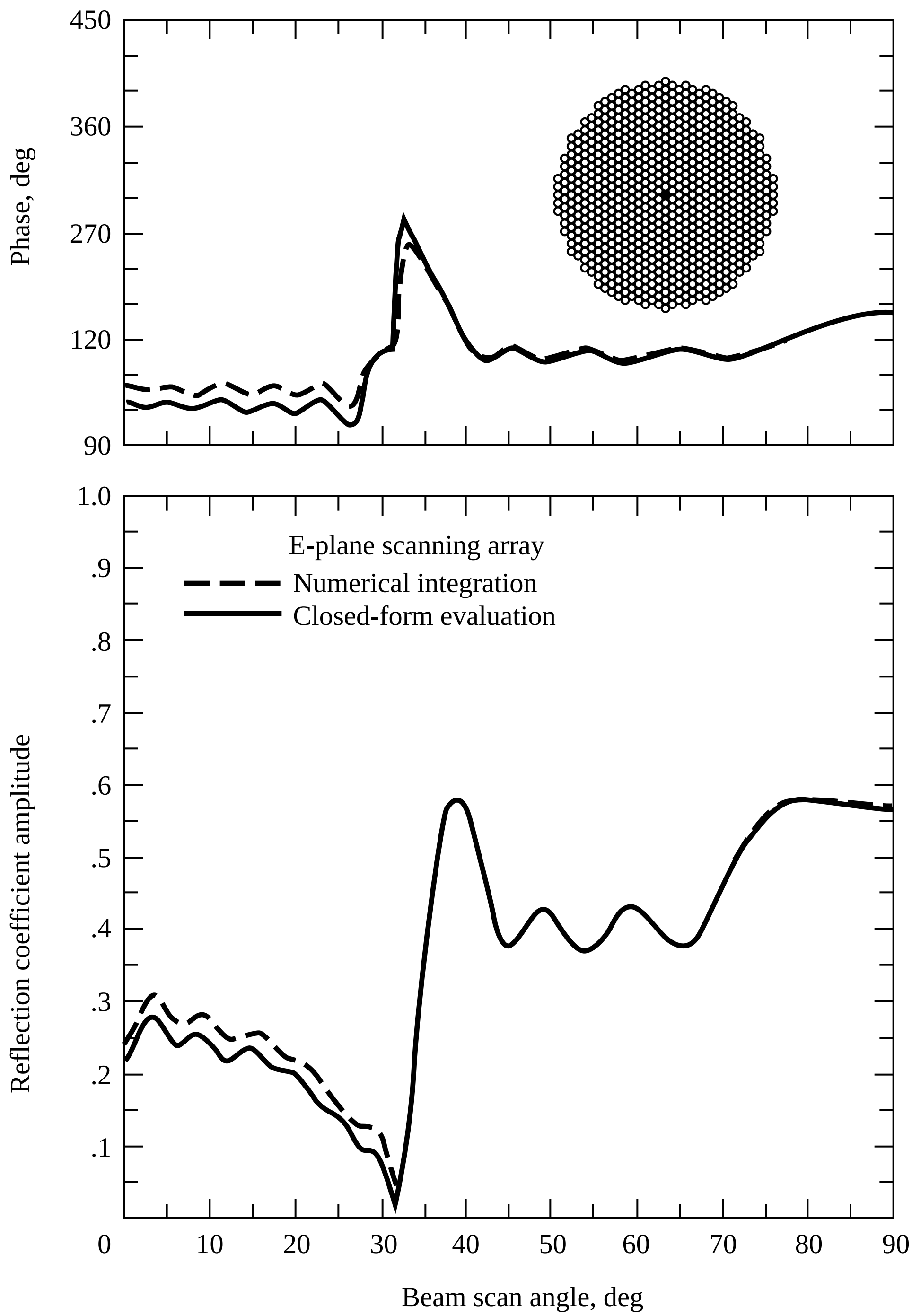

Figure 6. Reflection coefficient versus E-plane scan angle for center element of 721-element array.

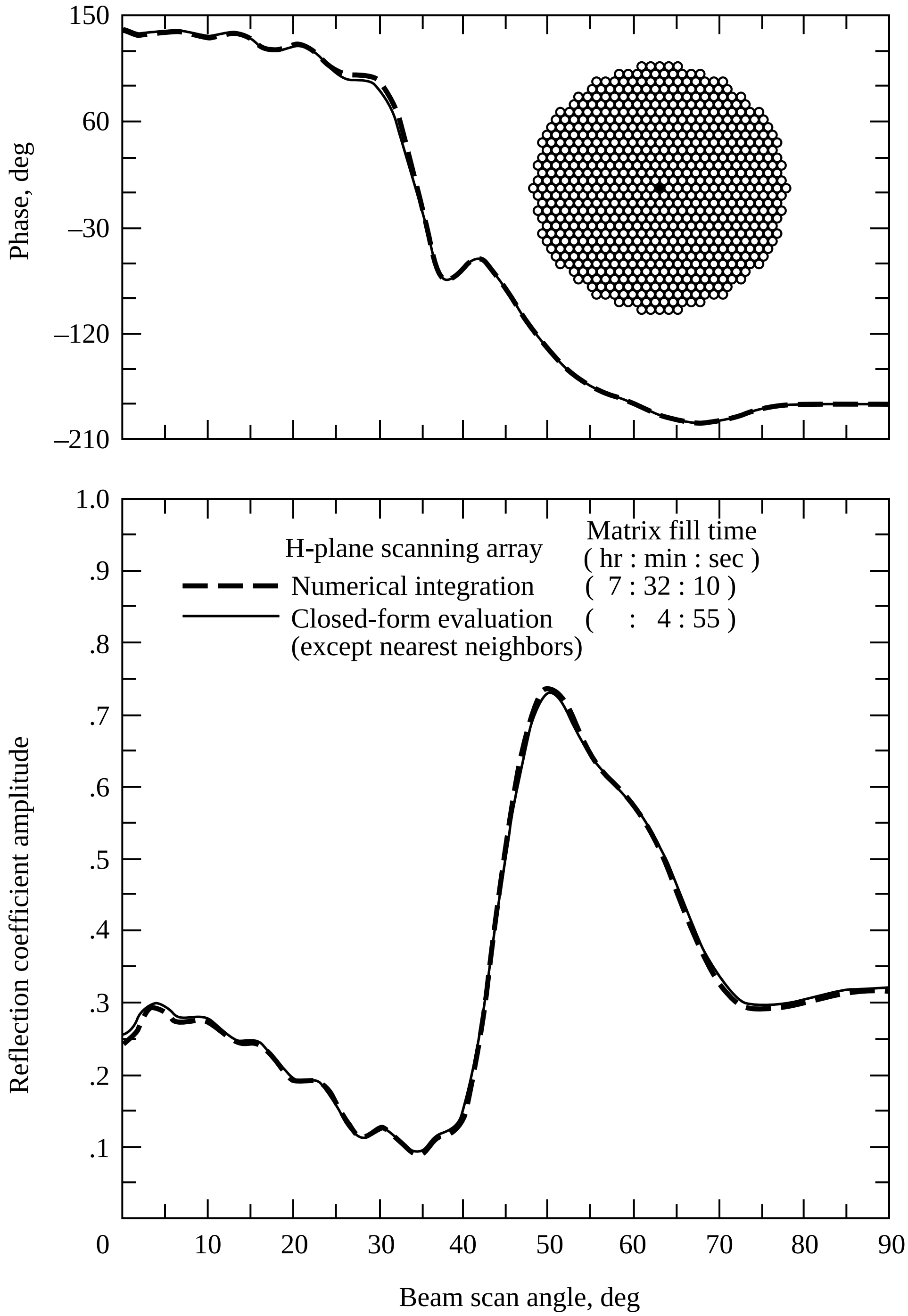

Figure 7. Reflection coefficient versus H-plane scan angle for center element of 721-element array.

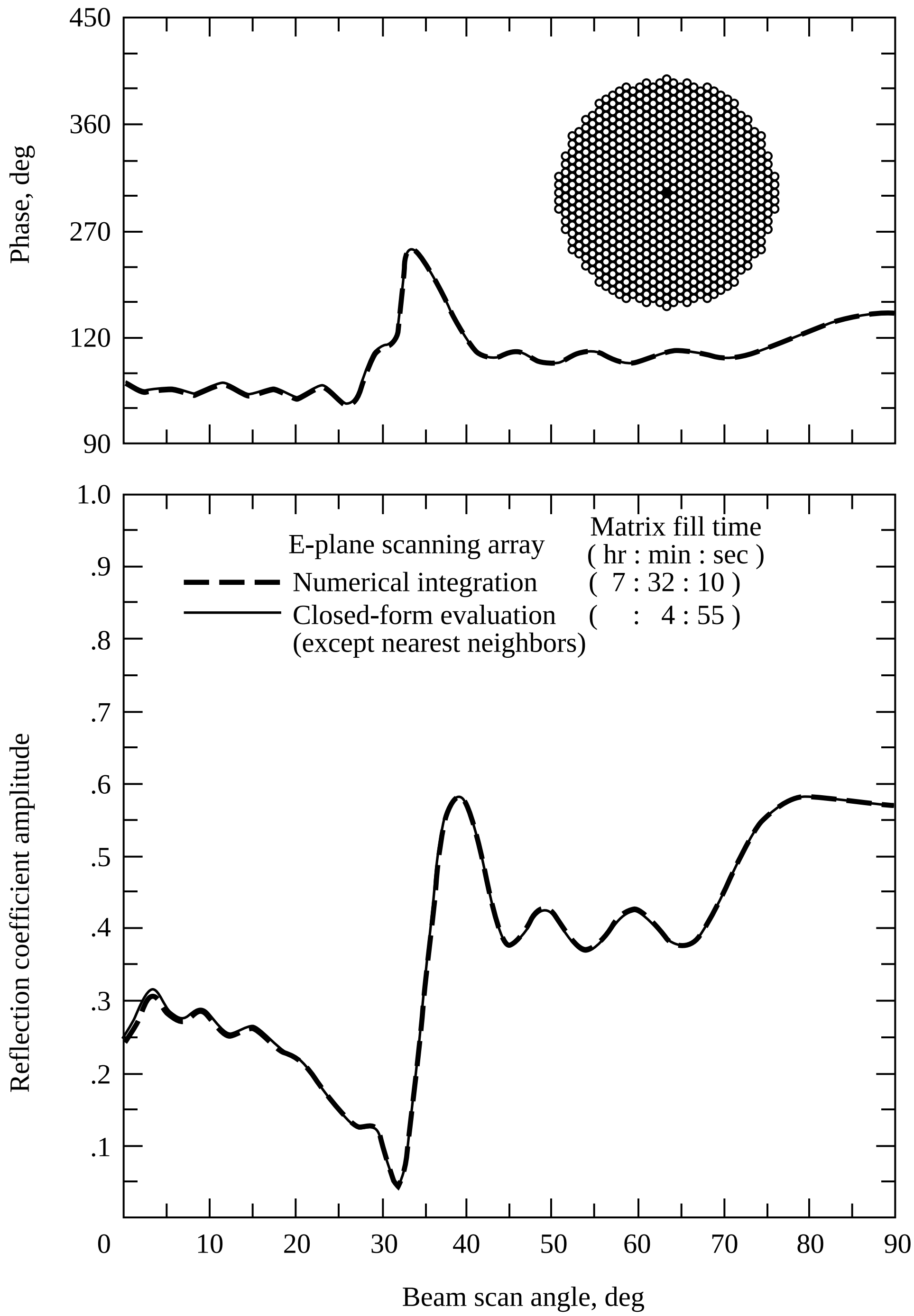

Figure 8. Reflection coefficient versus E-plane scan angle for center element of 721-element array.

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13. ABSTRACT *(Maximum 200 words)*
The integral expression for the mutual admittance between circular apertures in a planar array is evaluated in closed form. Very good accuracy is realized when compared with values that were obtained by numerical integration. Utilization of this closed-form expression, for all element pairs that are separated by more than one element spacing, yields extremely accurate results and significantly reduces the computation time that is required to analyze the performance of a large electronically scanning antenna array.

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