REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
June 1992

3. REPORT TYPE AND DATES COVERED
Technical Paper

4. TITLE AND SUBTITLE
Identification of Linear Systems by an Asymptotically Stable Observer

5. FUNDING NUMBERS
WU 590-14-61-01

6. AUTHOR(S)
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NASA Langley Research Center
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8. PERFORMING ORGANIZATION REPORT NUMBER
L-16940

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA TP-3164

11. SUPPLEMENTARY NOTES
Phan: Lockheed Engineering & Sciences Co., Hampton, VA, former National Research Council research associate at Langley; Horta and Juang: Langley Research Center, Hampton, VA; Longman: Columbia University, NY, NY, former National Research Council research associate at Langley.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified–Unlimited

Subject Category 39

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
This paper presents a formulation for the identification of a linear multivariable system from single or multiple sets of input-output data. The system input-output relationship is expressed in terms of an observer, which is made asymptotically stable by an embedded eigenvalue assignment procedure. The prescribed eigenvalues for the observer may be real, complex, mixed real and complex, or zero. In this formulation, the Markov parameters of the observer are identified from input-output data. The Markov parameters of the actual system are then recovered from those of the observer and used to obtain a state space model of the system by standard realization techniques. The basic mathematical formulation is derived, and extensive numerical examples using simulated noise-free data are presented to illustrate the proposed method.

14. SUBJECT TERMS
System identification; Markov parameters; Observer Markov parameters; Observer pole placement; System realization

15. NUMBER OF PAGES
67

16. PRICE CODE
A04

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified		

NSN 7540-01-280-5500

Standard Form 298(Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

NASA-Langley, 1992

Abstract

This paper presents a formulation for the identification of linear multivariable systems from single or multiple sets of input-output data. The system input-output relationship is expressed in terms of an observer, which is made asymptotically stable by an embedded eigenvalue assignment procedure. The prescribed eigenvalues for the observer may be real, complex, mixed real and complex, or zero. In this formulation, the Markov parameters of the observer are identified from input-output data. The Markov parameters of the actual system are then recovered from those of the observer and used to obtain a state space model of the system by standard realization techniques. The basic mathematical formulation is derived, and extensive numerical examples using simulated noise-free data are presented to illustrate the proposed method.

Nomenclature

Symbol	Dimensions	Description of symbol
A	$n \times n$	system matrix (discrete)
$\overline{A}$	$n \times n$	observer system matrix (discrete)
A_o	$n \times n$	system matrix in observable canonical form
$A(q^{-1})$		polynomial of delay operators associated with output
B	$n \times m$	system input influence matrix (discrete)
$\overline{B}$	$n \times (m+q)$	observer input influence matrix
B'	$n \times m$	partition of $\overline{B}$ associated with input $u(i)$
B'_d	$n \times m$	partition of $\overline{B}$ associated with input $u(i)$ for a deadbeat observer
$B(q^{-1})$		polynomial of delay operators associated with input
B^*	$n \times m$	transformed partition B' of $\overline{B}$ for a MIMO system, $B^* = TB'$
b	$n \times 1$	system input influence vector for a SISO system
$\bar{b}$	$n \times 2$	observer input influence vector for a SISO system
b_o	$n \times 1$	system input influence vector for a SISO system in observable canonical form
b'	$n \times 1$	partition of $\bar{b}$ associated with scalar input
b^*	$n \times 1$	transformed partition b' of $\bar{b}$ for a SISO system, $b^* = Tb'$
b_i^*	scalar	ith element of the transformed vector b^*
$b_{(i)}^{*T}$	$1 \times m$	ith row of the matrix B^*

C	$q \times n$	system output matrix
C^*	$q \times n$	transformed system output matrix for a MIMO system, $C^* = CT^{-1}$
c	$1 \times n$	system output vector for a SISO system
c_o	$1 \times n$	system output vector for a SISO system in observable canonical form
c^*	$1 \times n$	transformed output vector for SISO systems, $c^* = cT^{-1}$
c_i^*	scalar	ith element of the transformed output vector c^*
$c_{(i)}^*$	$q \times 1$	ith column of the transformed system output matrix C^*
D	$q \times m$	direct transmission matrix
d	scalar	direct transmission term for SISO systems
E_m	$sm \times m$	a matrix of identities and null matrices, $E_m^T = \begin{bmatrix} I_{m\times m} & O_{m\times(s-1)m} \end{bmatrix}$
E_q	$rq \times q$	a matrix of identities and null matrices, $E_q^T = \begin{bmatrix} I_{q\times q} & O_{q\times(r-1)q} \end{bmatrix}$
$H(\tau)$	$qr \times ms$	an $r \times s$ block data matrix of Markov parameters for realization
I		identity matrix
M	$n \times q$	observer gain
M_d	$n \times q$	deadbeat observer gain for a MIMO system
M^*	$n \times q$	transformed observer gain for a MIMO system, $M^* = TM$
$m_{(i)}^{*T}$	$1 \times q$	ith row of the matrix M^*
m	scalar or	number of inputs or
	$n \times 1$	observer gain for a SISO system
m_o	$n \times 1$	observer gain for a SISO system in observable canonical form
m_o^d	$n \times 1$	deadbeat observer gain for a SISO system in observable canonical form
m_i^d	scalar	ith element of observer gain m_o^d in observable canonical form
m^*	$n \times 1$	transformed observer gain for a SISO system, $m^* = Tm$
m_i^*	scalar	ith element of the transformed vector m^*
n	scalar	order or assumed order of a linear system

n_c	scalar	number of prescribed complex conjugate pairs of eigenvalues
n_r	scalar	number of prescribed real eigenvalues
O		null matrix
p	scalar	number of observer Markov parameters, $C\overline{B}$, $C\overline{A}\overline{B}$, ..., $C\overline{A}^{p-1}\overline{B}$, also referred to as window width in the identification algorithm
p_i	scalar	coefficients of characteristic equation for a SISO system
q	scalar	number of outputs
q^{-1}		one-time step delay operator
R^n, R^q, R^m		space of n-, q-, and m-dimensional real-valued vectors
r	scalar	pole radius of prescribed eigenvalues in the complex plane
T	$n \times n$	a similarity transformation matrix for $\overline{A}$, $\overline{A} = T^{-1}\Lambda T$
U, V	$rq \times n, sm \times n$	orthonormal matrices obtained from the singular value decomposition of the Hankel matrix $H(0)$
$u(i)$	$m \times 1$	input to system at time step i
$\underline{u}(i-p)$	$mp \times 1$	p-time step input history vector
$\underline{u}(i-n)$	$mn \times 1$	n-time step input history vector
$v(i)$	$(m+q) \times 1$	vector containing input and output served as "input" to observer at time step i
$x(i)$	$n \times 1$	system state vector at time step i
$\widehat{x}(i)$	$n \times 1$	estimated system state vector at time step i
$\widetilde{x}(i)$	$n \times 1$	state estimation error vector at time step i
Y_τ	$q \times m$	system Markov parameter, $Y_\tau = CA^\tau B$
$\overline{Y}_\tau$	$q \times (m+q)$	observer Markov parameter, $\overline{Y}_\tau = C\overline{A}^\tau \overline{B}$
$\overline{Y}_\tau^{(1)}$	$q \times m$	partition of $\overline{Y}_\tau$ associated with input $u(i)$
$\overline{Y}_\tau^{(2)}$	$q \times q$	partition of $\overline{Y}_\tau$ associated with output $y(i)$
$y(i)$	$q \times 1$	output of system at time step i
$\widehat{y}(i)$	$q \times 1$	estimated output at time step i
$\underline{y}(i-n)$	$qn \times 1$	n-time step output history vector
$\underline{y}(i-p)$	$qp \times 1$	p-time step output history vector
α	$n \times 1$	SISO observer parameter vector associated with input for real eigenvalue assignment

α_c	$n \times 1$	counterpart of α for SISO complex eigenvalue assignment
α_d	$n \times 1$	counterpart of α for SISO deadbeat eigenvalue assignment
α_m	$n \times 1$	counterpart of α for SISO mixed eigenvalue assignment
$\underline{\alpha}$	$q \times mn$	MIMO observer parameter matrix associated with input for real eigenvalue assignment
$\underline{\alpha_c}$	$q \times mn$	counterpart of $\underline{\alpha}$ for MIMO complex eigenvalue assignment
$\underline{\alpha_d}$	$q \times mn$	counterpart of $\underline{\alpha}$ for MIMO deadbeat eigenvalue assignment
$\underline{\alpha_m}$	$q \times mn$	counterpart of $\underline{\alpha}$ for MIMO mixed eigenvalue assignment
β	$n \times 1$	SISO observer parameter vector associated with output for real eigenvalue assignment
β_c	$n \times 1$	counterpart of β for SISO complex eigenvalue assignment
β_d	$n \times 1$	counterpart of β for SISO deadbeat eigenvalue assignment
β_m	$n \times 1$	counterpart of β for SISO mixed eigenvalue assignment
$\underline{\beta}$	$q \times qn$	MIMO observer parameter matrix associated with output for real eigenvalue assignment
$\underline{\beta_c}$	$q \times qn$	counterpart of $\underline{\beta}$ for MIMO complex eigenvalue assignment
$\underline{\beta_d}$	$q \times qn$	counterpart of $\underline{\beta}$ for MIMO deadbeat eigenvalue assignment
$\underline{\beta_m}$	$q \times qn$	counterpart of $\underline{\beta}$ for MIMO mixed eigenvalue assignment
$,(i-1)$	$(2n+1) \times 1$	combined vector of $\phi(i-1)$, $\varphi(i-1)$, and $u(i)$ for SISO real eigenvalue assignment
$,_c(i-1)$	$(2n+1) \times 1$	counterpart of $,(i-1)$ for SISO complex eigenvalue assignment
$,_d(i-1)$	$(2n+1) \times 1$	counterpart of $,(i-1)$ for SISO deadbeat eigenvalue assignment
$,_m(i-1)$	$(2n+1) \times 1$	counterpart of $,(i-1)$ for SISO mixed eigenvalue assignment
$\underline{,}(i-1)$	$(m+q)n+m \times 1$	combined vector of $\underline{\phi}(i-1)$, $\underline{\varphi}(i-1)$, and $u(i)$ for MIMO real eigenvalue assignment
$\underline{,_c}(i-1)$	$(m+q)n+m \times 1$	counterpart of $\underline{,}(i-1)$ for MIMO complex eigenvalue assignment

$\underline{\Gamma_d}(i-1)$	$(m+q)n+m \times 1$	counterpart of $\underline{\Gamma}(i-1)$ for MIMO deadbeat eigenvalue assignment
$\underline{\Gamma_m}(i-1)$	$(m+q)n+m \times 1$	counterpart of $\underline{\Gamma}(i-1)$ for MIMO mixed eigenvalue assignment
γ	$(2n+1) \times 1$	vector of combined observer parameters for SISO real eigenvalue assignment
γ_c	$(2n+1) \times 1$	counterpart of γ for SISO complex eigenvalue assignment
γ_d	$(2n+1) \times 1$	counterpart of γ for SISO deadbeat eigenvalue assignment
γ_m	$(2n+1) \times 1$	counterpart of γ for SISO mixed eigenvalue assignment
$\widehat{\gamma}(i)$	$(2n+1) \times 1$	vector of combined observer parameters estimated at time step i for SISO real eigenvalue assignment
$\widehat{\gamma}_c(i)$	$(2n+1) \times 1$	counterpart of $\widehat{\gamma}(i)$ for SISO complex eigenvalue assignment
$\widehat{\gamma}_d(i)$	$(2n+1) \times 1$	counterpart of $\widehat{\gamma}(i)$ for SISO deadbeat eigenvalue assignment
$\widehat{\gamma}_m(i)$	$(2n+1) \times 1$	counterpart of $\widehat{\gamma}(i)$ for SISO mixed eigenvalue assignment
$\underline{\gamma}$	$q \times (m+q)n+m$	matrix of combined observer parameters for MIMO real eigenvalue assignment
$\underline{\gamma_c}$	$q \times (m+q)n+m$	counterpart of $\underline{\gamma}$ for MIMO complex eigenvalue assignment
$\underline{\gamma_d}$	$q \times (m+q)n+m$	counterpart of $\underline{\gamma}$ for MIMO deadbeat eigenvalue assignment
$\underline{\gamma_m}$	$q \times (m+q)n+m$	counterpart of $\underline{\gamma}$ for MIMO mixed eigenvalue assignment
$\underline{\widehat{\gamma}}(i)$	$q \times (m+q)n+m$	matrix of combined observer parameters estimated at time step i for MIMO real eigenvalue assignment
$\underline{\widehat{\gamma}_c}(i)$	$q \times (m+q)n+m$	counterpart of $\underline{\widehat{\gamma}}(i)$ for MIMO complex eigenvalue assignment
$\underline{\widehat{\gamma}_d}(i)$	$q \times (m+q)n+m$	counterpart of $\underline{\widehat{\gamma}}(i)$ for MIMO deadbeat eigenvalue assignment
$\underline{\widehat{\gamma}_m}(i)$	$q \times (m+q)n+m$	counterpart of $\underline{\widehat{\gamma}}(i)$ for MIMO mixed eigenvalue assignment
Λ	$n \times n$	diagonal matrix of prescribed real eigenvalues
Λ_c	$n \times n$	block diagonal matrix of prescribed complex eigenvalues
Λ_m	$n \times n$	counterpart of Λ for mixed real and complex eigenvalues

λ_i		prescribed real or complex eigenvalues
$\lambda^{(\tau)}$	$n \times 1$	vector of powers of prescribed real eigenvalues for a SISO system
$\lambda_c^{(\tau)}$	$n \times 1$	counterpart of $\lambda^{(\tau)}$ for SISO prescribed complex eigenvalues
$\lambda_m^{(\tau)}$	$n \times 1$	counterpart of $\lambda^{(\tau)}$ for SISO prescribed mixed eigenvalues
$\underline{\lambda}_{i,m}^{(\tau)}$	$m \times m$	diagonal matrix of τ-powers of real eigenvalue λ_i repeated m times
$\underline{\lambda}_{i,q}^{(\tau)}$	$q \times q$	diagonal matrix of τ-powers of real eigenvalue λ_i repeated q times
$\underline{\lambda}_m^{(\tau)}$	$mn \times m$	matrix of powers of prescribed real eigenvalues associated with input for a MIMO system
$\underline{\lambda}_{c,m}^{(\tau)}$	$mn \times m$	counterpart of $\underline{\lambda}_m^{(\tau)}$ associated with input for MIMO complex eigenvalue assignment
$\underline{\lambda}_{m,m}^{(\tau)}$	$mn \times m$	counterpart of $\underline{\lambda}_m^{(\tau)}$ associated with input for MIMO mixed eigenvalue assignment
$\underline{\lambda}_q^{(\tau)}$	$qn \times q$	matrix of powers of prescribed real eigenvalues associated with output for a MIMO system
$\underline{\lambda}_{c,q}^{(\tau)}$	$qn \times q$	counterpart of $\underline{\lambda}_q^{(\tau)}$ associated with output for MIMO complex eigenvalue assignment
$\underline{\lambda}_{m,q}^{(\tau)}$	$qn \times q$	counterpart of $\underline{\lambda}_q^{(\tau)}$ associated with output for MIMO mixed eigenvalue assignment
Σ	$n \times n$	diagonal matrix of positive singular values
σ_i	scalar	real part of a complex eigenvalue
$\sigma_i^{(\tau)}$, $\omega_i^{(\tau)}$	scalars	elements of the matrix $\begin{bmatrix} \sigma_i^{(\tau)} & \omega_i^{(\tau)} \\ -\omega_i^{(\tau)} & \sigma_i^{(\tau)} \end{bmatrix} \equiv \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}^{\tau}$
$\underline{\sigma}_{i,m}^{(\tau)}$	$m \times m$	diagonal matrix of $\sigma_i^{(\tau)}$ repeated m times
$\underline{\sigma}_{i,q}^{(\tau)}$	$q \times q$	diagonal matrix of $\sigma_i^{(\tau)}$ repeated q times
$\phi(i-1)$	$n \times 1$	transformed p-time step input history vector $\phi(i-1) = \Im \underline{u}(i-p)$ for SISO real eigenvalue assignment
$\phi_c(i-1)$	$n \times 1$	counterpart of $\phi(i-1)$ for SISO complex eigenvalue assignment

$\phi_m(i-1)$	$n \times 1$	counterpart of $\phi(i-1)$ for SISO mixed eigenvalue assignment
$\underline{\phi}(i-1)$	$nm \times 1$	transformed p-time step input history vector $\underline{\phi}(i-1) = \underline{\Im}_m \underline{u}(i-p)$ for MIMO real eigenvalue assignment
$\underline{\phi_c}(i-1)$	$nm \times 1$	counterpart of $\underline{\phi}(i-1)$ for MIMO complex eigenvalue assignment
$\underline{\phi_m}(i-1)$	$nm \times 1$	counterpart of $\underline{\phi}(i-1)$ for MIMO mixed eigenvalue assignment
$\varphi(i-1)$	$n \times 1$	transformed p-time step output history vector $\varphi(i-1) = \Im \underline{y}(i-p)$ for SISO real eigenvalue assignment
$\varphi_c(i-1)$	$n \times 1$	counterpart of $\varphi(i-1)$ for SISO complex eigenvalue assignment
$\varphi_m(i-1)$	$n \times 1$	counterpart of $\varphi(i-1)$ for SISO mixed eigenvalue assignment
$\underline{\varphi}(i-1)$	$nq \times 1$	transformed p-time step output history vector $\underline{\varphi}(i-1) = \underline{\Im}_q \underline{y}(i-p)$ for MIMO real eigenvalue assignment
$\underline{\varphi_c}(i-1)$	$nq \times 1$	counterpart of $\underline{\varphi}(i-1)$ for MIMO complex eigenvalue assignment
$\underline{\varphi_m}(i-1)$	$nq \times 1$	counterpart of $\underline{\varphi}(i-1)$ for MIMO mixed eigenvalue assignment
ω_i	scalar	complex part of a complex eigenvalue
$\underline{\omega}_{i,m}^{(\tau)}$	$m \times m$	diagonal matrix of $\omega_i^{(\tau)}$ repeated m times
$\underline{\omega}_{i,q}^{(\tau)}$	$q \times q$	diagonal matrix of $\omega_i^{(\tau)}$ repeated q times
$\Im$	$n \times p$	Vandermonde-like matrix of real eigenvalues for SISO systems
$\Im_c$	$n \times p$	counterpart of $\Im$ for SISO complex eigenvalue assignment
$\Im_m$	$n \times p$	counterpart of $\Im$ for SISO mixed eigenvalue assignment
$\underline{\Im}_m$	$mn \times mp$	Vandermonde-like matrix of real eigenvalues associated with input for a MIMO system
$\underline{\Im}_{c,m}$	$mn \times mp$	counterpart of $\underline{\Im}_m$ for MIMO complex eigenvalue assignment
$\underline{\Im}_{m,m}$	$mn \times mp$	counterpart of $\underline{\Im}_m$ for MIMO mixed eigenvalue assignment
$\underline{\Im}_q$	$qn \times qp$	Vandermonde-like matrix of real eigenvalues associated with output for a MIMO system
$\underline{\Im}_{c,q}$	$qn \times qp$	counterpart of $\underline{\Im}_q$ for MIMO complex eigenvalue assignment

$\underline{\Im}_{m,q}$	$qn \times qp$	counterpart of $\underline{\Im}_q$ for MIMO mixed eigenvalue assignment
$\Re(i)$	$(m+q)n+m$ $\times(m+q)n+m$	projection or variance matrix for recursive least-squares estimation

Abbreviations:

ARMA	auto-regressive moving average
ERA	Eigensystem Realization Algorithm
MIMO	multiple-input multiple-output
SISO	single-input single-output

Introduction

The aim of learning identification is to provide methods to improve identification of the system model as additional information about the system becomes available. The techniques are in the time domain, and the system information comes in the form of input-output data from either multiple experiments or a single experiment of extended duration. Originally, the idea of learning identification was motivated by the fact that for system identification of flexible structures, multiple experiments are often performed with the hope that the averaged data can reduce the effects of irregularities such as measurement noises, repetitive disturbances, and slight nonlinearities. This motivates the development of learning identification to improve identification results effectively from multiple experiments. An early technique for identification of parameters from multiple experiments was formulated in reference 1. Learning identification in the present form identifies the Markov parameters from general input-output data (ref. 2). The Markov parameters are then used to obtain a state space model of the system by a realization procedure, e.g., the Eigensystem Realization Algorithm (ERA) (refs. 3 and 4). The learning identification procedures presented in reference 2 require input-output data from a large number of experiments of generally short duration. The procedures identify as many Markov parameters as the number of data samples in each experiment, and the number of data samples that can be used is constrained by the number of Markov parameters desired to be identified. In practice, there may be substantially more data samples in each experiment than the number of desired Markov parameters. Therefore, these techniques are not efficient in the sense that they do not necessarily make use of all available input-output data. This motivates the development of identification algorithms from a single set of input-output data of extended duration. Learning identification is closely related in concept and technique to learning control, where the motivation is to develop control laws that improve tracking error based on repeated execution of a task (refs. 5–11).

An identification procedure from a single set of input-output data is developed in reference 12 by means of an auto-regressive moving average (ARMA) description of the original system in state space format via an observer. An important distinguishing feature of the approach presented in reference 12 as opposed to previous development is that the system is identified indirectly by an observer, which is made asymptotically stable by an eigenvalue assignment procedure. The discrete-time eigenvalues are required to be real, distinct, with magnitudes less than one. The recursive formulation of reference 12 extends the repetition domain concept used in learning control and identification to shifting time intervals. It is based on procedures that identify system Markov parameters for indirect learning control and repetitive control (refs. 9 and 10). The use of Markov parameters in system identification is discussed in reference 13. In this paper, the identification technique is generalized to allow assignment of complex eigenvalues.

This generalization is particularly important when the order of the system is large, since it permits a more even distribution of asymptotically stable eigenvalues inside the unit circle by using the entire complex plane.

The basic contributions of this paper are as follows: First, a simplified reformulation of the original identification algorithm with placement of real eigenvalues is presented. Second, the formulation is extended to the case of complex eigenvalue assignment, which is also applicable to the general case of assigning both real and complex eigenvalues. Third, a version of this identification procedure using a deadbeat observer with poles placed at the origin is formulated. This case is of particular interest since it makes use of a minimum number of data samples, and the number of identified observer Markov parameters are reduced to a minimum set. Fourth, an extensive numerical study is provided to illustrate the basic characteristics of the algorithm. The deterministic technique developed here is applicable for data from either a single set or multiple sets of experiments. Because of the complexities in the formulation of the identification algorithms, the case of single-input single-output systems will be first described. The results are then extended to the case of multiple-input multiple-output systems. This paper gives a more detailed presentation of the results in reference 14, with additional examples. In order to study the exact nature of the identification procedure under ideal circumstances, this paper is confined to purely deterministic results. In the presence of process and measurement noises, the relationship between the identification algorithm with a deadbeat observer presented in this paper and the stochastic Kalman filter algorithm of reference 15 is established in reference 16. A procedure to improve observer and Kalman filter identification results by whitening the residual sequence is presented in reference 17. Often of interest in practice is the identification of a model in a prescribed frequency range. Such a development of the algorithm is formulated in reference 18.

The general outline of the paper is as follows. The procedure with real eigenvalue assignment, which is first presented in reference 12, is reformulated here using a modified mathematical formulation. The modified formulation allows direct extension of the procedure to the case with complex eigenvalue assignment. A special case of the identification procedure when all eigenvalues are placed at the origin is then presented. For clarity, the formulation for single-input single-output systems is presented in the main body of the paper, except for the case of mixed real and complex eigenvalues assignment, which is presented in appendix A. Extensions of the identification procedure to multiple-input multiple-output systems are parallel to the developments for the single-input single-output case. The multivariable case is presented in appendix B. The truss structure used in the numerical example section is described in appendix C.

Mathematical Preliminaries

The following general mathematical formulation is applicable to both single-input single-output (SISO) and multiple-input multiple-output (MIMO) systems. This section introduces the basic concepts and establishes some general mathematical relations, which are used to derive the identification algorithm in subsequent sections.

System Description

In this section, the relationship between the state space model and a particular auto-regressive moving average (ARMA) model of a linear system is presented. This relationship is particularly useful for developing an identification procedure. First, consider a general discrete multivariable linear system expressed in the state space format

$$\left.\begin{aligned} x(i+1) &= Ax(i) + Bu(i) \\ y(i) &= Cx(i) + Du(i) \end{aligned}\right\} \tag{1}$$

where $x(i) \in R^n, y(i) \in R^q, u(i) \in R^m$. Let $x(0)$ denote the initial state at $i = 0$. An input-output description of the above system can be obtained from equation (1) as

$$y(i) = CA^i x(0) + \sum_{\tau=0}^{i-1} CA^{i-\tau-1}Bu(\tau) + Du(i) \tag{2}$$

Note that the first term on the right-hand side of the above equation is dependent on the initial condition $x(0)$. The products $CA^{i-\tau-1}B$ denoted by $Y_{i-\tau-1}$, together with D, are known as the Markov parameters of the system. From equation (2), the input-output description of the system with zero initial conditions becomes

$$y(i) = \sum_{\tau=0}^{i-1} Y_\tau u(i-\tau-1) + Du(i) \tag{3}$$

where $y(i)$ is expressed in terms of Y_0 up to Y_{i-1} and the direct transmission term D. In general, this description requires $i+1$ Markov parameters to describe the output at time step i. If the system is asymptotically stable such that the Markov parameters Y_p, Y_{p+1}, Y_{p+2}, ... can be neglected for some p, then at time steps $i \geq p$, the input-output description can be approximated with a finite set of Markov parameters as

$$y(i) \approx \sum_{\tau=0}^{p-1} Y_\tau u(i-\tau-1) + Du(i) \tag{4}$$

It is important to note that for a finite dimensional system, there is only a finite number of independent system Markov parameters. Therefore, the system Markov parameters used in the description of equation (4) are not necessarily independent. For sufficiently damped systems, equation (4) is a valid description of the input-output relationship provided that p is chosen sufficiently large such that the approximation holds. However, for lightly damped systems, such as large flexible space structures, the ARMA model would require a very large number of Markov parameters, which would not be computationally attractive for system identification. In fact, if the system is unstable or marginally stable, such a description is no longer possible. In the following, a procedure is described to express the state space model in equation (1) as an ARMA model with a finite number of Markov parameters. The Markov parameters can be shown to be those of an observer system that is made asymptotically stable by eigenvalue assignment. This observer model is then used to develop an identification method for the system described by equation (1).

To construct an observer model, add and subtract the term $My(i)$ to the right-hand side of the state equation in equation (1) to yield

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i) + My(i) - My(i) \\ &= (A+MC)x(i) + (B+MD)u(i) - My(i) \end{aligned} \tag{5}$$

Define

$$\left.\begin{aligned} \overline{A} &= A + MC \\ \overline{B} &= [B + MD, -M] \\ v(i) &= \begin{bmatrix} u(i) \\ y(i) \end{bmatrix} \end{aligned}\right\} \tag{6}$$

Then the original system becomes

$$\left.\begin{aligned} x(i+1) &= \overline{A}x(i) + \overline{B}v(i) \\ y(i) &= Cx(i) + Du(i) \end{aligned}\right\} \tag{7}$$

The input-output description of the system with zero initial conditions is

$$y(i) = \sum_{\tau=0}^{i-1} \overline{Y}_{i-\tau-1} v(\tau) + Du(i) \tag{8}$$

where

$$\overline{Y}_{i-\tau-1} = C\overline{A}^{\,i-\tau-1}\overline{B}$$

If the system is made asymptotically stable by placing of the eigenvalues of the matrix $\overline{A}$ such that the Markov parameters $\overline{Y}_p$, $\overline{Y}_{p+1}$, $\overline{Y}_{p+2}$, $\ldots$ can be neglected for some p, then at time steps $i \geq p$, the input-output description can be approximated with a reduced set of $p+1$ Markov parameters $\left\{\overline{Y}_0,\ \overline{Y}_1,\ \ldots,\ \overline{Y}_{p-1},\ D\right\}$. The following equality

$$y(i) = \sum_{\tau=0}^{p-1} \overline{Y}_\tau v(i-\tau-1) + Du(i) \qquad (i \geq p) \tag{9}$$

then approximately holds. If the original system is observable, then for any system matrix A, it is always possible to find a matrix M such that the desired eigenvalues of $\overline{A}$ are placed in any particular (symmetric) configuration. For the case of lightly damped systems, this procedure can transform the set of an otherwise large number of Markov parameters to an approximately equivalent reduced set $\left\{\overline{Y}_0,\ \overline{Y}_1,\ \ldots,\ \overline{Y}_{p-1},\ D\right\}$ by selecting appropriate eigenvalues for $\overline{A}$. Furthermore, for a sufficiently large p, the influence of a nonzero initial condition on the output at time steps $i \geq p$ can be neglected. The model of equation (9) is used to develop the identification method presented herein, and the eigenvalue assignment step is achieved implicitly through processing of the measured input-output data. To see that equation (9) is a special auto-regressive moving average model, it can be rewritten as

$$y(i) + \sum_{\tau=0}^{p-1} C(A+MC)^\tau My(i-\tau-1) = \sum_{\tau=0}^{p-1} C(A+MC)^\tau (B+MD)u(i-\tau-1) + Du(i) \tag{10}$$

Defining a delay operator q^{-1} applied to a variable $z(i)$ to be $q^{-1}z(i) \equiv z(i-1)$, the above equation can be written in the usual deterministic ARMA model format

$$A\left(q^{-1}\right) y(i) = B\left(q^{-1}\right) u(i) \tag{11}$$

with the polynomials of the delay operators $A\left(q^{-1}\right)$ and $B\left(q^{-1}\right)$ given as

$$A\left(q^{-1}\right) = I + CMq^{-1} + C\overline{A}Mq^{-2} + \cdots + C\overline{A}^{p-1}Mq^{-p}$$

$$B\left(q^{-1}\right) = D + CB'q^{-1} + C\overline{A}B'q^{-2} + \cdots + C\overline{A}^{p-1}B'q^{-p}$$

where $\overline{A} = A + MC$ and $B' = B + MD$.

Relations of the System to an Observer Model

The role of the matrix M in the above development can be interpreted in terms of an observer model. Consider the system given in equation (1). It has an observer of the form

$$\begin{aligned} \widehat{x}(i+1) &= A\widehat{x}(i) + Bu(i) - M[y(i) - \widehat{y}(i)] \\ \widehat{y}(i) &= C\widehat{x}(i) + Du(i) \end{aligned} \tag{12}$$

It can be shown from equations (12) and (1) that

$$\begin{aligned} \widehat{x}(i+1) &= A\widehat{x}(i) + Bu(i) - MC[x(i) - \widehat{x}(i)] \\ &= (A + MC)\widehat{x}(i) + Bu(i) - M[y(i) - Du(i)] \\ &= (A + MC)\widehat{x}(i) + (B + MD)u(i) - My(i) \end{aligned} \tag{13}$$

Defining the state estimation error $\widetilde{x}(i) = x(i) - \widehat{x}(i)$, the equation that governs $\widetilde{x}(i)$ is

$$\begin{aligned} \widetilde{x}(i+1) &= Ax(i) + Bu(i) - [(A + MC)\widehat{x}(i) + (B + MD)u(i) - My(i)] \\ &= (A + MC)\widetilde{x}(i) \end{aligned} \tag{14}$$

If system (1) is observable, then M may be chosen to place the eigenvalues of $A + MC$ in any desired (symmetric) configuration. In particular, they will be placed inside the unit circle in the complex plane. From equation (14), if M is chosen such that $A + MC$ is asymptotically stable, then $\lim_{i\to\infty} \widetilde{x}(i) = 0$; i.e., the estimated state $\widehat{x}(i)$ converges to the true state $x(i)$ as i approaches infinity. Equation (13) then becomes

$$x(i+1) = (A + MC)x(i) + (B + MD)u(i) - My(i) \tag{15}$$

which is exactly the same as equation (5).

From this analysis, matrix M can be interpreted as an observer gain. The parameters $\overline{Y}_{i-\tau-1} = C\overline{A}^{i-\tau-1}\overline{B}$ in equation (8) are then the Markov parameters of an observer system; hence they are now referred to as observer Markov parameters. In the identification process, these are the parameters to be identified. Once they are identified, the actual system Markov parameters can be recovered. There is an algebraic relationship between the Markov parameters of the observer system and those of the actual system. This result is established in the following section.

Relations Between the Markov Parameters of the Observer and the Actual System

As before, let the Markov parameters of the observer system be denoted by $\overline{Y}_\tau$, and the Markov parameters of the actual system by Y_τ. Recall that

$$
\begin{aligned}
\overline{Y}_\tau &= C\overline{A}^\tau\overline{B} \\
&= [C(A+MC)^\tau(B+MD), -C(A+MC)^\tau M] \\
&\equiv [\overline{Y}_\tau^{(1)}, \overline{Y}_\tau^{(2)}]
\end{aligned} \tag{16}
$$

From the second equation in equation (16), the Markov parameter CB of the system is simply

$$
\begin{aligned}
Y_0 = CB &= C(B+MD) - (CM)D \\
&= \overline{Y}_0^{(1)} + \overline{Y}_0^{(2)}D
\end{aligned} \tag{17}
$$

To obtain the Markov parameter CAB, first consider the product $\overline{Y}_1{}^{(1)}$

$$
\begin{aligned}
\overline{Y}_1^{(1)} &= C(A+MC)(B+MD) \\
&= CAB + CMCB + C(A+MC)MD
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Y_1 &= CAB \\
&= \overline{Y}_1^{(1)} + \overline{Y}_0^{(2)}Y_0 + \overline{Y}_1^{(2)}D
\end{aligned} \tag{18}
$$

Similarly, to obtain the Markov parameter CA^2B, consider the product $\overline{Y}_2^{(1)}$

$$
\begin{aligned}
\overline{Y}_2^{(1)} &= C(A+MC)^2(B+MD) \\
&= C(A^2 + MCA + AMC + MCMC)(B+MD) \\
&= CA^2B + CMCAB + C(A+MC)MCB + C(A+MC)^2MD
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Y_2 &= CA^2B \\
&= \overline{Y}_2^{(1)} - CMCAB - C(A+MC)MCB - C(A+MC)^2MD \\
&= \overline{Y}_2^{(1)} + \overline{Y}_0^{(2)}Y_1 + \overline{Y}_1^{(2)}Y_0 + \overline{Y}_2^{(2)}D
\end{aligned} \tag{19}
$$

By induction, the general relationship between the actual system Markov parameters and the observer Markov parameters can be shown to be

$$
Y_\tau = \overline{Y}_\tau^{(1)} + \sum_{i=0}^{\tau-1} \overline{Y}_i^{(2)} Y_{\tau-i-1} + \overline{Y}_\tau^{(2)} D \tag{20}
$$

For a noise-free finite-dimensional system, knowledge of a sufficient number of actual system Markov parameters is adequate to deduce a state space realization of the system of interest. Physical aspects of the model such as natural frequencies, damping ratios, and mode shapes can then be found.

Identification Theory for Single-Input Single-Output Systems

In the following, an identification method is developed to identify the coefficients of an ARMA model that is made asymptotically stable by an embedded eigenvalue assignment procedure. The coefficients of the ARMA model are precisely the observer Markov parameters formulated in the above section. For simplicity, consider the case of a single-input single-output system in the state space format:

$$\begin{aligned} x(i+1) &= Ax(i) + bu(i) \\ y(i) &= cx(i) + du(i) \end{aligned} \tag{21}$$

where $x(i) \in R^n$, and $u(i)$ and $y(i)$ are scalars. The system matrix A is an $n \times n$ matrix, b an $n \times 1$ column vector, c a $1 \times n$ row vector, and the direct transmission term d is a scalar. The input-output description of this system is given as in equation (9):

$$y(i) = \sum_{\tau=0}^{p-1} \overline{Y}_\tau v(i - \tau - 1) + du(i) \tag{22}$$

where

$$\overline{Y}_\tau = c\overline{A}^\tau \overline{b} = c(A + mc)^\tau [b + md, -m] = [c\overline{A}^\tau b', -c\overline{A}^\tau m] \tag{23}$$

The observer gain m in this case is an $n \times 1$ column vector. Recall that $v(i)$ contains both the input $u(i)$ and the output $y(i)$. For $i \geq p$, equation (22) can be rewritten as an approximate ARMA model:

$$y(i) = \sum_{\tau=0}^{p-1} (c\overline{A}^\tau b')\, u\,(i - \tau - 1) + du(i) - \sum_{\tau=0}^{p-1} (c\overline{A}^\tau m)\, y\,(i - \tau - 1) \tag{24}$$

Derived in the following section is an algorithm that computes the coefficients $c\overline{A}^\tau b'$ and $c\overline{A}^\tau m$ of the ARMA model in equation (24) and simultaneously places the eigenvalues of $\overline{A}$ in prescribed locations so as to make the ARMA model asymptotically stable. These eigenvalues may be real, complex conjugate pairs, a combination of both, or zero (deadbeat).

SISO Real Eigenvalue Assignment

This is the simplest case, where all the prescribed eigenvalues are real and distinct. The eigenvalue assignment procedure can be derived by noting that for desired real and distinct eigenvalues of $\overline{A}$, one has for some nonsingular matrix T

$$\overline{A} = T^{-1} \Lambda T \tag{25}$$

where Λ is a diagonal matrix of n prescribed eigenvalues,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \tag{26}$$

For simplicity, the blank spaces denote zero elements. Then the products $c\overline{A}^{\tau}b'$ and $c\overline{A}^{\tau}m$ become

$$c\overline{A}^{\tau}b' = cT^{-1}\Lambda^{\tau}Tb' \qquad c\overline{A}^{\tau}m = cT^{-1}\Lambda^{\tau}Tm \tag{27}$$

If the elements of $c^* \equiv cT^{-1}$, $b^* \equiv Tb'$, and $m^* \equiv Tm$ are written explicitly as

$$c^* = \begin{bmatrix} c_1^* & c_2^* & \cdots & c_n^* \end{bmatrix} \qquad b^* = \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_n^* \end{bmatrix} \qquad m^* = \begin{bmatrix} m_1^* \\ m_2^* \\ \vdots \\ m_n^* \end{bmatrix} \tag{28}$$

then the product $c\overline{A}^{\tau}b'$ in equation (24) may be expressed as

$$c\overline{A}^{\tau}b' = c^*\Lambda^{\tau}b^* = \begin{bmatrix} c_1^*b_1^* & c_2^*b_2^* & \cdots & c_n^*b_n^* \end{bmatrix} \begin{bmatrix} \lambda_1^{\tau} \\ \lambda_2^{\tau} \\ \vdots \\ \lambda_n^{\tau} \end{bmatrix} \tag{29}$$

Similarly,

$$-c\overline{A}^{\tau}m = -c^*\Lambda^{\tau}m^* = \begin{bmatrix} -c_1^*m_1^* & -c_2^*m_2^* & \cdots & -c_n^*m_n^* \end{bmatrix} \begin{bmatrix} \lambda_1^{\tau} \\ \lambda_2^{\tau} \\ \vdots \\ \lambda_n^{\tau} \end{bmatrix} \tag{30}$$

With the following simplified notations

$$\left.\begin{aligned} \alpha^T &= \begin{bmatrix} c_1^*b_1^* & c_2^*b_2^* & \cdots & c_n^*b_n^* \end{bmatrix} \\ \beta^T &= \begin{bmatrix} -c_1^*m_1^* & -c_2^*m_2^* & \cdots & -c_n^*m_n^* \end{bmatrix} \\ \lambda^{(\tau)} &= \begin{bmatrix} \lambda_1^{\tau} & \lambda_2^{\tau} & \cdots & \lambda_n^{\tau} \end{bmatrix}^T \end{aligned}\right\} \tag{31}$$

equation (24) becomes

$$y(i) = \alpha^T \sum_{\tau=0}^{p-1} \lambda^{(\tau)} u(i-\tau-1) + \beta^T \sum_{\tau=0}^{p-1} \lambda^{(\tau)} y(i-\tau-1) + du(i) \tag{32}$$

or simply

$$y(i) = \gamma^T, (i-1) \tag{33}$$

which is in linear form with the unknown parameters in α^T, β^T, and d with

$$\gamma^T = \begin{bmatrix} \alpha^T & \beta^T & d \end{bmatrix} \qquad , (i-1) = \begin{bmatrix} \phi(i-1) \\ \varphi(i-1) \\ u(i) \end{bmatrix} \tag{34}$$

where

$$\left.\begin{aligned} \phi(i-1) &= \sum_{\tau=0}^{p-1} \lambda^{(\tau)} u(i-\tau-1) = \Im \underline{u}(i-p) \\ \varphi(i-1) &= \sum_{\tau=0}^{p-1} \lambda^{(\tau)} y(i-\tau-1) = \Im \underline{y}(i-p) \end{aligned}\right\} \tag{35}$$

The matrix $\Im$ is a Vandermonde-like matrix of prescribed real eigenvalues of magnitudes less than 1:

$$\Im = \begin{bmatrix} \lambda_1^{p-1} & \lambda_1^{p-2} & \cdots & \lambda_1 & 1 \\ \lambda_2^{p-1} & \lambda_2^{p-2} & \cdots & \lambda_2 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_n^{p-1} & \lambda_n^{p-2} & \cdots & \lambda_n & 1 \end{bmatrix} \tag{36}$$

and the $p \times 1$ input and output history vectors $\underline{u}(i-p)$ and $\underline{y}(i-p)$ are defined as

$$\underline{u}(i-p) = \begin{bmatrix} u(i-p) \\ \vdots \\ u(i-2) \\ u(i-1) \end{bmatrix} \qquad \underline{y}(i-p) = \begin{bmatrix} y(i-p) \\ \vdots \\ y(i-2) \\ y(i-1) \end{bmatrix} \tag{37}$$

Note that equation (33) is in linear form; thus the unknown observer parameter vector γ can be solved for directly from input-output data. For on-line computation, however, recursive solution is often preferred. Let $\hat{\gamma}(i)$ denote the estimated parameter vector at time step i. The standard recursive least-squares solution to equation (33) is simply

$$\left.\begin{aligned} \hat{\gamma}(i) &= \hat{\gamma}(i-1) + \frac{\Re(i-2), (i-1)}{1 + , (i-1)^T \Re(i-2), (i-1)} \left[y(i) - \hat{\gamma}^T(i-1), (i-1) \right] \\ \Re(i-1) &= \Re(i-2) - \frac{\Re(i-2), (i-1), (i-1)^T \Re(i-2)}{1 + , (i-1)^T \Re(i-2), (i-1)} \end{aligned}\right\} \tag{38}$$

with an arbitrary initial guess $\hat{\gamma}(0)$, and $\Re(-1)$ is any positive definite matrix. Other recursive parameter estimation algorithms may be used to replace the standard least squares at this step, e.g., the projection or instrumental variable methods (refs. 19 and 20). The above algorithm identifies the parameter vector γ, which consists of the products $c_i^* b_i^*$, $-c_i^* m_i^*$ $(i = 1, 2, \ldots, n)$, and d. These products together with the assigned eigenvalues λ_i $(i = 1,\ 2,\ \ldots,\ n)$ can be used to reconstruct the observer Markov parameters $\overline{Y}_\tau$ $(\tau = 0,\ 1,\ 2,\ \ldots)$ as

$$
\begin{aligned}
\overline{Y}_\tau &= c\overline{A}^{\,\tau}\,\overline{b} = c\overline{A}^{\,\tau}[b', -m] \\
&= \left[\begin{bmatrix} c_1^* b_1^* & c_2^* b_2^* & \cdots & c_n^* b_n^* \end{bmatrix} \begin{bmatrix} \lambda_1^\tau \\ \lambda_2^\tau \\ \vdots \\ \lambda_n^\tau \end{bmatrix}, \begin{bmatrix} -c_1^* m_1^* & -c_2^* m_2^* & \cdots & -c_n^* m_n^* \end{bmatrix} \begin{bmatrix} \lambda_1^\tau \\ \lambda_2^\tau \\ \vdots \\ \lambda_n^\tau \end{bmatrix} \right] \\
&= \begin{bmatrix} \alpha^T \lambda^{(\tau)} & \beta^T \lambda^{(\tau)} \end{bmatrix}
\end{aligned}
\tag{39}
$$

Finally, the actual system Markov parameters can then be recovered from the above reconstructed observer Markov parameters according to equation (20):

$$
\begin{aligned}
Y_\tau &= \overline{Y}_\tau^{(1)} + \sum_{i=0}^{\tau-1} \overline{Y}_\tau^{(2)} Y_{\tau-i-1} + \overline{Y}_\tau^{(2)} D \\
&= \alpha^T \lambda^{(\tau)} + \beta^T \left(\sum_{i=0}^{\tau-1} \lambda^{(\tau)} Y_{\tau-i-1} + \lambda^{(\tau)} d \right)
\end{aligned}
\tag{40}
$$

where $\overline{Y}_\tau^{(1)} = \alpha^T \lambda^{(\tau)}$ and $\overline{Y}_\tau^{(2)} = \beta^T \lambda^{(\tau)}$.

SISO Complex Eigenvalue Assignment

With the general mathematical framework developed for real eigenvalue assignment, the procedure for complex eigenvalue assignment can be similarly derived by replacing equations (25) and (26) with their counterparts for complex conjugate pairs of eigenvalues $\lambda_i = \sigma_i \pm j\omega_i$ $(i = 1,\ 2,\ \ldots,\ n/2)$. Namely, $\overline{A} = T^{-1}\Lambda_c T$, and

$$
\Lambda_c = \begin{bmatrix} \begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix} & & & \\ & \begin{bmatrix} \sigma_2 & \omega_2 \\ -\omega_2 & \sigma_2 \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} \sigma_{n/2} & \omega_{n/2} \\ -\omega_{n/2} & \sigma_{n/2} \end{bmatrix} \end{bmatrix}
\tag{41}
$$

Furthermore, associated with the complex conjugate pair $\lambda_i = \sigma_i \pm j\omega_i$, write

$$\begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}^{\tau} \equiv \begin{bmatrix} \sigma_i^{(\tau)} & \omega_i^{(\tau)} \\ -\omega_i^{(\tau)} & \sigma_i^{(\tau)} \end{bmatrix} \tag{42}$$

then the product $c\overline{A}^{\,\tau}b'$ in equation (24) becomes

$$\begin{aligned} c\overline{A}^{\,\tau}b' &= c^*\Lambda_c^\tau b^* \\ &= \sigma_1^{(\tau)}(c_1^*b_1^* + c_2^*b_2^*) + \omega_1^{(\tau)}(c_1^*b_2^* - c_2^*b_1^*) + \sigma_2^{(\tau)}(c_3^*b_3^* + c_4^*b_4^*) + \omega_2^{(\tau)}(c_3^*b_4^* - c_4^*b_3^*) \\ &\quad + \cdots + \sigma_{n/2}^{(\tau)}\left(c_{n-1}^*b_{n-1}^* + c_n^*b_n^*\right) + \omega_{n/2}^{(\tau)}\left(c_{n-1}^*b_n^* - c_n^*b_{n-1}^*\right) \\ &= \alpha_c^T\lambda_c^{(\tau)} \end{aligned} \tag{43}$$

Similarly,

$$\begin{aligned} c\overline{A}^{\,\tau}m &= c^*\Lambda_c^\tau m^* \\ &= \sigma_1^{(\tau)}(c_1^*m_1^* + c_2^*m_2^*) + \omega_1^{(\tau)}(c_1^*m_2^* - c_2^*m_1^*) + \sigma_2^{(\tau)}(c_3^*m_3^* + c_4^*m_4^*) + \omega_2^{(\tau)}(c_3^*m_4^* - c_4^*m_3^*) \\ &\quad + \cdots + \sigma_{n/2}^{(\tau)}\left(c_{n-1}^*m_{n-1}^* + c_n^*m_n^*\right) + \omega_{n/2}^{(\tau)}\left(c_{n-1}^*m_n^* - c_n^*m_{n-1}^*\right) \\ &= -\beta_c^T\lambda_c^{(\tau)} \end{aligned} \tag{44}$$

where

$$\left.\begin{aligned} \alpha_c^T &= \begin{bmatrix} c_1^*b_1^* + c_2^*b_2^* & c_1^*b_2^* - c_2^*b_1^* & \cdots & c_{n-1}^*b_{n-1}^* + c_n^*b_n^* & c_{n-1}^*b_n^* - c_n^*b_{n-1}^* \end{bmatrix} \\ \beta_c^T &= -\begin{bmatrix} c_1^*m_1^* + c_2^*m_2^* & c_1^*m_2^* - c_2^*m_1^* & \cdots & c_{n-1}^*m_{n-1}^* + c_n^*m_n^* & c_{n-1}^*m_n^* - c_n^*m_{n-1}^* \end{bmatrix} \\ \lambda_c^{(\tau)} &= \begin{bmatrix} \sigma_1^{(\tau)} & \omega_1^{(\tau)} & \sigma_2^{(\tau)} & \omega_2^{(\tau)} & \cdots & \sigma_{n/2}^{(\tau)} & \omega_{n/2}^{(\tau)} \end{bmatrix}^T \qquad \left(\sigma_i^{(0)} = 1, \quad \omega_i^{(0)} = 0\right) \end{aligned}\right\} \tag{45}$$

The elements c_i^*, b_i^*, and m_i^* ($i = 1, 2, \ldots, n$) are defined exactly the same way as in equations (28), and n is now necessarily even, since all the prescribed eigenvalues must appear as complex conjugate pairs. Equation (24) now becomes

$$\begin{aligned} y(i) &= \alpha_c^T\sum_{\tau=0}^{p-1}\lambda_c^{(\tau)}u(i-\tau-1) + \beta_c^T\sum_{\tau=0}^{p-1}\lambda_c^{(\tau)}y(i-\tau-1) + du(i) \\ &= \gamma_c^T, {}_c(i-1) \end{aligned} \tag{46}$$

which is in linear form, with

$$\gamma_c^T = \begin{bmatrix} \alpha_c^T & \beta_c^T & d \end{bmatrix} \qquad , _c(i-1) = \begin{bmatrix} \phi_c(i-1) \\ \varphi_c(i-1) \\ u(i) \end{bmatrix} \tag{47}$$

where

$$\left.\begin{aligned} \phi_c(i-1) &= \sum_{\tau=0}^{p-1} \lambda_c^{(\tau)} u(i-\tau-1) = \Im_c \underline{u}(i-p) \\ \varphi_c(i-1) &= \sum_{\tau=0}^{p-1} \lambda_c^{(\tau)} y(i-\tau-1) = \Im_c \underline{y}(i-p) \end{aligned}\right\} \tag{48}$$

The matrix $\Im_c$ is a Vandermonde-like matrix of prescribed complex eigenvalues of magnitudes less than unity

$$\Im_c = \begin{bmatrix} \sigma_1^{(p-1)} & \sigma_1^{(p-2)} & \cdots & \cdots & \sigma_1 & 1 \\ \omega_1^{(p-1)} & \omega_1^{(p-2)} & \cdots & \cdots & \omega_1 & 0 \\ \sigma_2^{(p-2)} & \sigma_2^{(p-2)} & \cdots & \cdots & \sigma_2 & 1 \\ \omega_2^{(p-1)} & \omega_2^{(p-2)} & \cdots & \cdots & \omega_2 & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \sigma_{n/2}^{(p-1)} & \sigma_{n/2}^{(p-2)} & \cdots & \cdots & \sigma_{n/2} & 1 \\ \omega_{n/2}^{(p-1)} & \omega_{n/2}^{(p-2)} & \cdots & \cdots & \omega_{n/2} & 0 \end{bmatrix} \tag{49}$$

and the $p\times1$ input and output history vectors $\underline{u}(i-p)$ and $\underline{y}(i-p)$ are defined as in equations (37). Let $\widehat{\gamma}_c(i)$ denote the estimated parameter vector at time step i. The recursive least-squares solution for the complex eigenvalue case is obtained by simply replacing $\widehat{\gamma}(i)$ by $\widehat{\gamma}_c(i)$, $,\ (i-1)$ by $,\ _c(i-1)$ in equations (38) with an arbitrary initial guess $\widehat{\gamma}_c(0)$ given, and $\Re(-1)$ is any positive definite matrix $\Re_0$. Any other recursive algorithm may be used to replace the standard least squares at this step. The algorithm identifies the parameter vector γ_c, which consists of the product sums and differences $c_{i-1}^* b_{i-1}^* + c_i^* b_i^*$, $c_{i-1}^* b_i^* - c_i^* b_{i-1}^*$, $c_{i-1}^* m_{i-1}^* + c_i^* m_i^*$, $c_{i-1}^* m_i^* - c_i^* m_{i-1}^*$ $(i = 1,\ 2,\ \ldots,\ n)$, and d. These identified parameters, together with the assigned conjugate pairs of complex eigenvalues λ_i $(i = 1,\ 2,\ \ldots,\ n/2)$, can be used to reconstruct the observer system Markov parameters $\overline{Y}_\tau$ $(\tau = 0,\ 1,\ 2,\ \ldots)$

$$\begin{aligned} \overline{Y}_\tau &= c\overline{A}^{\,\tau}\,\overline{b} = c\overline{A}^{\,\tau}\,[b', -m] \\ &= \begin{bmatrix} \alpha_c^T \lambda_c^{(\tau)} & \beta_c^T \lambda_c^{(\tau)} \end{bmatrix} \equiv \begin{bmatrix} \overline{Y}_\tau^{(1)} & \overline{Y}_\tau^{(2)} \end{bmatrix} \end{aligned} \tag{50}$$

Finally, the actual system Markov parameters can then be recovered from the above reconstructed observer Markov parameters according to equation (20) in the same way as the real eigenvalue case:

$$Y_\tau = \alpha_c^T \lambda^{(\tau)} + \beta_c^T \left(\sum_{i=0}^{\tau-1} \lambda_c^{(\tau)} Y_{\tau-i-1} + \lambda_c^{(\tau)} d \right) \tag{51}$$

SISO Deadbeat Eigenvalue Assignment

If all the eigenvalues of the deterministic observer system are placed at the origin, then the Markov parameters of the observer system will become identically zero after a finite number of time steps. This is a deadbeat observer. Specifically,

$$\overline{Y}_\tau \equiv 0 \qquad (\tau = n,\ n+1,\ n+2,\ \ldots) \tag{52}$$

where n is the order of the system. Let m_d denote the deadbeat observer gain, the expression relating the input-output of the system and the corresponding observer Markov parameter is given by

$$y(i) = \sum_{\tau=0}^{n-1} \left(c\overline{A}^\tau b' \right) u(i-\tau-1) - \sum_{\tau=0}^{n-1} \left(c\overline{A}^\tau m_d \right) y(i-\tau-1) + du(i) \tag{53}$$

The structure of $\overline{A}$ can be better seen by considering the system given in equation (21) in observable canonical form:

$$A_o = \begin{bmatrix} 0 & & & & -a_1 \\ 1 & 0 & & & -a_2 \\ & 1 & \ddots & & -a_3 \\ & & \ddots & 0 & \vdots \\ & & & 1 & -a_n \end{bmatrix} \qquad b_o = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \qquad c_o = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{54}$$

Let the observer gain in observable canonical form be denoted by $m_o = [m_1\ m_2\ m_3\ \cdots\ m_n]^T$. The observer system matrix $\overline{A} = A_o + m_o c_o$ is simply

$$\overline{A} = \begin{bmatrix} 0 & & & & -a_1 + m_1 \\ 1 & 0 & & & -a_2 + m_2 \\ & 1 & 0 & & -a_3 + m_3 \\ & & 1 & \ddots & \vdots \\ & & & 1 & -a_n + m_n \end{bmatrix} \tag{55}$$

For a prescribed set of eigenvalues λ_i for $\overline{A}$, the observer gain m_o is unique and its elements are given by $m_i = a_i - p_i$, where p_i are the coefficients of the characteristic equation

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \lambda^n + p_n \lambda^{n-1} + \cdots + p_2 \lambda + p_1 = 0 \tag{56}$$

Let $m_o^d = [\, m_1^d \quad m_2^d \quad \cdots \quad m_n^d \,]^T$ denote the deadbeat observer gain for the system in observable canonical form. In the deadbeat case, the characteristic equation is simply $\lambda^n = 0$. Hence, $p_i = 0$,

$m_i^d = a_i$. The observer system matrix $\overline{A}$ then becomes

$$\overline{A} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & \ddots & \\ & & & 1 & 0 \end{bmatrix} \tag{57}$$

In this case, it is convenient to work with the system in observable canonical form directly. The Markov parameters of the deadbeat observer system can be computed as

$$\left.\begin{aligned} \overline{Y}_0 &= c_o\overline{b}_o = c_o\begin{bmatrix} b_o + m_o^d d & -m_o^d \end{bmatrix} = \begin{bmatrix} b_n + m_n^d d & -m_n^d \end{bmatrix} \\ \overline{Y}_1 &= c_o\overline{A}\,\overline{b}_o = \begin{bmatrix} b_{n-1} + m_{n-1}^d d & -m_{n-1}^d \end{bmatrix} \\ &\vdots \\ \overline{Y}_{n-1} &= c_o\overline{A}^{\,n-1}\,\overline{b}_o = \begin{bmatrix} b_1 + m_1^d d & -m_1^d \end{bmatrix} \\ \overline{Y}_n &= \overline{Y}_{n+1} = \overline{Y}_{n+2}\cdots = 0 \end{aligned}\right\} \tag{58}$$

Equation (24) becomes

$$y(i) = \sum_{\tau=0}^{n-1}\left(b_{n-\tau} + m_{n-\tau}^d d\right)u(i-\tau-1) - \sum_{\tau=0}^{n-1} m_{n-\tau}^d y(i-\tau-1) + du(i) \tag{59}$$

Defining the parameter vectors

$$\left.\begin{aligned} \alpha_d^T &= \begin{bmatrix} b_n + m_n^d d & b_{n-1} + m_{n-1}^d d & \cdots & b_1 + m_1^d d \end{bmatrix} \\ &= \begin{bmatrix} \overline{Y}_0^{(1)} & \overline{Y}_1^{(1)} & \cdots & \overline{Y}_{n-1}^{(1)} \end{bmatrix} \\ \beta_d^T &= \begin{bmatrix} -m_n^d & -m_{n-1}^d & \cdots & -m_1^d \end{bmatrix} \\ &= \begin{bmatrix} \overline{Y}_0^{(2)} & \overline{Y}_1^{(2)} & \cdots & \overline{Y}_{n-1}^{(2)} \end{bmatrix} \end{aligned}\right\} \tag{60}$$

equation (59) can then be written as

$$\begin{aligned} y(i) &= \alpha_d^T\,\underline{u}(i-n) + \beta_d^T\,\underline{y}(i-n) + du(i) \\ &= \gamma_d^T\,,\,{}_d(i-1) \end{aligned} \tag{61}$$

where

$$\gamma_d^T = \begin{bmatrix} \alpha_d^T & \beta_d^T & d \end{bmatrix} \qquad , _d(i-1) = \begin{bmatrix} \underline{u}(i-n) \\ \underline{y}(i-n) \\ u(i) \end{bmatrix} \tag{62}$$

and the $n \times 1$ input and output history vectors $\underline{u}(i-n)$ and $\underline{y}(i-n)$ are defined as

$$\underline{u}(i-n) = \begin{bmatrix} u(i-n) \\ \vdots \\ u(i-2) \\ u(i-1) \end{bmatrix} \qquad \underline{y}(i-n) = \begin{bmatrix} y(i-n) \\ \vdots \\ y(i-2) \\ y(i-1) \end{bmatrix} \tag{63}$$

The recursive solution to equation (61) is obtained by simply replacing $\widehat{\gamma}(i)$ by $\widehat{\gamma}_d(i)$, and , $(i-1)$ by , $_d(i-1)$ in equations (38). The actual system Markov parameters can be recovered according to equation (20) as

$$\begin{aligned} Y_\tau &= \overline{Y}_\tau^{(1)} + \sum_{i=0}^{\tau-1} \overline{Y}_\tau^{(2)} Y_{\tau-i-1} + \overline{Y}_\tau^{(2)} d \\ &= \left(b_{n-\tau} + m_{n-\tau}^d d \right) - \sum_{i=0}^{n-1} m_{n-\tau}^d Y_{\tau-i-1} - m_{n-\tau}^d d \\ &= b_{n-\tau} - \sum_{i=0}^{n-1} m_{n-\tau}^d Y_{\tau-i-1} \end{aligned} \tag{64}$$

where $b_{n-\tau} = m_{n-\tau}^d \equiv 0$, for $\tau = n,\ n+1, \ldots$.

A particular feature of the deadbeat algorithm is that the observer system Markov parameters are identically zero after a finite number of time steps. The input-output ARMA relation given in equation (22) or equation (24) used in deriving the algorithm therefore holds exactly. This is different from the previous cases, where by placing real and complex eigenvalues of magnitudes less than unity but greater than zero, the ARMA relation only holds approximately. The degree to which the approximation holds depends on the choices of prescribed eigenvalues and the window width p, i.e., the number of observer Markov parameters retained to maintain a valid approximation. In the deadbeat case, however, the approximation becomes exact, the window width p is the order of the system, and the identified parameters contain an exact description of the system of interest.

Realization by the Eigensystem Realization Algorithm

A state space model of the system from the recovered Markov parameters can be obtained by the Eigensystem Realization Algorithm (ERA). The algorithm begins with an $r \times s$ block data matrix called the Hankel matrix and denoted by $H(\tau)$

$$H(\tau) = \begin{bmatrix} Y_\tau & Y_{\tau+1} & \cdots & Y_{\tau+s-1} \\ Y_{\tau+1} & Y_{\tau+2} & \cdots & Y_{\tau+s} \\ \vdots & \vdots & \cdots & \vdots \\ Y_{\tau+r-1} & Y_{\tau+r} & \cdots & Y_{\tau+r+s-2} \end{bmatrix} \tag{65}$$

The order of the system is determined by the singular value decomposition of $H(0)$,

$$H(0) = U\Sigma V^T \tag{66}$$

where the columns of U and V are orthonormal, Σ is an $n \times n$ diagonal matrix of positive singular values, and n is the order of the system. Defining a $q \times rq$ matrix E_q^T, and an $m \times sm$ matrix E_m^T made up of identity and null matrices of the form

$$E_q^T = \begin{bmatrix} I_{q\times q} & O_{q\times(r-1)q} \end{bmatrix} \qquad E_m^T = \begin{bmatrix} I_{m\times m} & O_{m\times(s-1)m} \end{bmatrix} \tag{67}$$

a discrete-time minimal order realization of the system can be shown to be

$$\left.\begin{aligned} A_r &= \Sigma^{-1/2} U^T H(1) V \Sigma^{-1/2} \\ B_r &= \Sigma^{1/2} V^T E_m \\ C_r &= E_q^T U \Sigma^{1/2} \end{aligned}\right\} \tag{68}$$

This is the basic ERA formulation. To use ERA in the present identification procedure, the entries that make up the data matrix given in equation (65) are precisely the recovered system Markov parameters Y_τ ($\tau = 0,\ 1,\ 2,\ \ldots$). For further details on the algorithm, the readers are referred to various references in the literature, e.g., references 3 and 4.

Computation Steps

This section reviews the basic steps involved to implement the identification procedure developed in this paper. The related equations are identified in each step of the process.

Step 1

Assume an order n for the system to be identified. Choose an order p for the ARMA model, and select the prescribed eigenvalues of the observer. For the eigenvalue assignment procedures, p is normally several times larger than the assumed order of the system, n. Specifically, the value of p chosen must be consistent with the prescribed eigenvalues for the observer, as described in the following:

(a) For real eigenvalues, select n real eigenvalues λ_i ($i = 1,\ 2,\ \ldots,\ n$) such that $\lambda_i^p \approx 0$.

(b) For complex eigenvalues, the eigenvalues must appear in complex conjugate pairs, $\lambda_i = \sigma_i \pm j\omega_i$ $(i = 1,\ 2,\ \ldots,\ n/2)$ such that

$$\begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}^p \approx 0$$

(c) For a combination of real and complex eigenvalues, the same rules apply.

(d) For deadbeat observers, however, all eigenvalues are set to be zero, and p is the same as n. The identification equations for the deadbeat case have taken this into account. Therefore, no explicit specification of the eigenvalues for this case is necessary.

Note that for asymptotic stability, all prescribed real or complex eigenvalues must have magnitudes less than unity.

Step 2

Compute the observer parameters. The appropriate recursive equations used for each case are outlined as follows. For observers with assigned real eigenvalues, equations (38) are used for the SISO case, and equations (B13) are used for the MIMO case. For observers with complex eigenvalues, the recursive equations are obtained simply by replacing $\hat{\gamma}(i)$ by $\hat{\gamma}_c(i)$, and $,\,(i-1)$ by $,_c(i-1)$ in equations (38) for the SISO case, and by replacing $\underline{\hat{\gamma}}$ by $\underline{\hat{\gamma}_c}$, and $\underline{,}(i-1)$ by $\underline{,_c}(i-1)$ in equations (B13) for the MIMO case. For observers with mixed real and complex eigenvalues, replace $\hat{\gamma}(i)$ by $\hat{\gamma}_m(i)$, and $,\,(i-1)$ by $,_m(i-1)$ in equations (38) for the SISO case, and $\hat{\gamma}(i)$ by $\underline{\hat{\gamma}_m}(i)$, and $,\,(i-1)$ by $\underline{,_m}(i-1)$ in equations (B13) for the MIMO case. For deadbeat observers, the appropriate recursive equations are obtained by replacing $\hat{\gamma}(i)$ by $\hat{\gamma}_d(i)$, and $,\,(i-1)$ by $,_d(i-1)$ in equations (38) for the SISO case, and $\hat{\gamma}(i)$ by $\underline{\hat{\gamma}_d}(i)$, and $,_d(i-1)$ by $\underline{,_d}(i-1)$ in equations (B13) for the MIMO case.

Step 3

Reconstruct the observer Markov parameters from the identified observer parameters. For observers with real eigenvalues, equation (39) is used for the SISO case, and equation (B14) is used for the MIMO case. For observers with complex eigenvalues, equation (50) and equation (B23) are used, respectively. Similarly, for observers with both real and complex eigenvalues, equation (A11) and equation (B31) are used. For deadbeat observers, however, the identified parameters are precisely the observer Markov parameters, and no reconstruction of the observer Markov parameters is needed for this case.

Step 4

Recover the system Markov parameters from the observer Markov parameters. The general equation is given in equation (20), which is then specialized to various cases. For observers with real eigenvalues, equation (40) is used for the SISO case, and equation (B15) is used for the MIMO case. For observers with complex eigenvalues, equation (51) is used for the SISO case, and equation (B24) is used for the MIMO case. For observers with both real and complex eigenvalues, equation (A12) and equation (B32) are used, respectively. For deadbeat observers, equation (20) directly applies.

Step 5

Realize a state space model for the identified system from the recovered system Markov parameters in step 4 above. The basic equations for ERA are summarized in equations (65) to (68).

Numerical Examples

The theoretical development sections discussed the use of observers and eigenvalue placement to recover the system Markov parameters. The Markov parameters are the pulse response samples of a linear system. The fundamental idea in the developed identification procedure is to identify parameters of an observer rather than those of the actual system. From the observer parameters the true system parameters can be recovered. The observer eigenvalues or poles determine the observer pulse response decay rate. Formulations where the prescribed eigenvalues are real, complex, mix real and complex, and zero (deadbeat) have been presented. By making the pulse response of the observer system decay sufficiently fast through the placement of its poles, one can truncate the response after a finite number of time steps. Because of the different eigenvalue placement procedures, this approximation will result in different convergence properties for each respective algorithm.

To study the numerical properties of the identification procedure, an analytical model of a truss structure is used. The lightly damped structure, known as the Mini-Mast (ref. 21) at NASA Langley Research Center, is modeled by its first five modes, with frequencies of 0.80, 0.80, 4.36, 6.10, and 6.16 Hz. A more detailed description of the system under consideration is given in appendix C. The outputs correspond to displacement sensors, and the inputs to torque actuators. The input-output data are simulated using random inputs for 6 sec. The system is discretized at a sampling rate of 33.3 Hz, and an input-output history of 200 points is recorded for system identification, which is performed on a Macintosh IIci computer. The analytical model contains five modes, but practically only three of them are controllable and observable from any given input-output pair.

Single-Input Single-Output Examples

First, for clarity the case of single-input single-output identification is studied. Basic characteristics of the identification algorithm can be seen in the SISO case. For this purpose, the first input second output pair is used for identification, which results in a system with essentially three identifiable structural modes, i.e., a sixth-order system. Results for the cases of real, complex, and deadbeat eigenvalue assignments are presented. The case of mixed real and complex eigenvalue assignment is omitted here since its numerical properties may be deduced from those of real and complex eigenvalue assignments.

Consider the case of real eigenvalue assignment. The identification results for this case are reported in figures 1(a)–1(d). Figure 1(a) shows the nominal case where six observer poles are placed at ± 0.2, ± 0.3, and ± 0.4. Along with the prescribed pole locations is an estimate of the number of samples or window width p that it takes for the observer pulse response to decay to a negligible value. In this example, the window width is selected to be 40 points wide, i.e., $p = 40$, so that $(\pm 0.2)^p$, $(\pm 0.3)^p$, $(\pm 0.4)^p$ are negligible. The identification procedure starts with an initial estimate of the system order, which for the nominal case the assumed order is six, $n = 6$. Even though the model used is of 10th-order, from any input-output pair the effective order of the system is only 6.

The top left plot of figure 1(a) shows convergence histories of the observer parameter values calculated from the recursive least-squares solution given in equations (38). The constant values correspond to converged parameters. Since the initial parameters are assumed to be zero, to start the algorithm, the projection matrix $\Re(-1)$ is set to a large value to reflect the degree of uncertainty of the initial guess. The plot on the top right shows the square root of the diagonal elements of the variance or projection matrix $\Re(i)$ after 160 iterations of the recursive least-squares algorithm. In cases where the exact least-squares solution is obtained and no order over-specification occurs, the variance matrix approaches zero. In general, the variance matrix provides a measure of the freedom in the uniqueness of the identified parameters. Note that when the identified parameters are not all independent because of order over-specification, the large variance values do not imply inaccuracies in the parameter estimates. This merely means that for the specified order, the identified set of observer parameters is not unique. The recursive least-squares solution is driven by the prediction error shown in the second row of figure 1(a). At any time step, the prediction error is defined to be the difference between the true output value and the predicted output value computed based on the estimated model available at that time step. The initial prediction error is large but quickly goes to zero as the observer parameters converge to constant values. For the case of real eigenvalue assignment, from the identified observer parameters, the observer Markov parameters are recovered by equation (39). The actual system Markov parameters are then computed by equation (40). Using the computed pulse response, realization of a state-space representation of the system is performed using equations (65)–(68). At this step, the initial assumption made about the system order ($n = 6$) is verified by counting the number of nonzero singular values. Shown in the second row of figure 1(a) is a bar chart of the normalized singular values which shows six nonzero singular values.

The top four plots of figure 1(a) are indicators as to how well the parameters are identified. The bottom four plots show results comparing the identified state space model and the true system model. Included in this group are comparisons of realized and actual pulse responses; actual displacement history used in the identification, and its reconstruction using the identified model; and the frequency response functions. There are two curves in each plot; the solid curve corresponds to actual data and the dashed curve to reconstruction. When an exact model of the system is identified, the two sets of curves overlap.

To study the effect of order under-specification, figure 1(b) shows the results when the observer poles are placed on the real axis, as in figure 1(a), but the assumed system order is set to $n = 2$. This is a case where not enough freedom is allowed in the identification procedure. The parameter values, shown on the top left of figure 1(b), do not tend to constant values as in figure 1(a). Although the variance is small, the prediction error shows discrepancies between the predicted and actual outputs. Realization using the identified parameters results in a system of order two, as shown by the singular value plot. When comparing the impulse responses, it is clear that the results are in error. So are the reconstructed displacement and frequency response functions. In this case, the algorithm attempts to identify a sixth-order system by a second-order model. Figure 1(c) shows the results when the assumed system order is increased to four, $n = 4$. Convergence of the parameters is observed, and the corresponding variance is small. The prediction error fluctuates about zero. The realized system order is four, as depicted in the singular value plot. Comparing the pulse responses shows very small errors. However, the frequency response functions show that the identified system (depicted by the dashed curve) missed the mode with the smallest contribution to the system response. This is why the reconstructed displacement, when compared with the actual displacement as shown in the lower left plot, shows no visible differences. This example suggests a potential application of the algorithm for identification of reduced order models.

To examine the case of order over-specification, figure 1(d) shows the results when the observer poles are also placed on the real axis as before, but the order of the system is over specified to be $n = 10$. Results are similar to those shown in figure 1(a), with two important distinctions. First, the parameter variances are now substantially larger than those in the previous cases; in fact they are an order of magnitude larger than the identified parameters. Second, the realized system order is correctly identified to be 6 even though the initial assumed order is 10. Large variances are expected when the identified parameters are not all linearly independent. When order over-specification occurs, there are more parameters than necessary to identify the system exactly. It is important to observe that at the realization step, however, the system and its order are identified correctly. For the case of SISO identification, if the assumed order is less than or equal to the true order of the system, as shown in figures 1(a)–1(c), the algorithm returns an identified model with the same order as assumed. However, if the assumed order is more than the true order, the algorithm returns a model with the correct minimal order, as shown in figure 1(d). The identification procedure, as mentioned earlier, places the observer poles at prescribed locations. To verify the proper eigenvalue placement, the observer pulse responses are used to realize the observer model, and the recovered eigenvalues are found to be identical to the prescribed values in all cases.

The next group of figures (figs. 2(a)–2(d)) presents results when complex poles are prescribed. Six complex poles are placed on a circle with a radius $r = 0.5$ in the complex plane corresponding to the same damping level. The window width is selected to be $p = 40$. The top left plot shows the parameter convergence histories. The identified parameters are now given by equation (45) instead of equations (31). The overall performance given in terms of prediction error, reconstructed response, pulse response, and frequency response functions is similar to that of the real case. Results for the complex case with order under-specification, $n = 2$, are shown in figure 2(b). When the assumed order is increased to four, figure 2(c) shows that the identified solution misses the weakest mode of the system. Again, this is consistent with previous results. In the complex case when the system order is over specified, $n = 10$, some of the parameters do not converge, as shown in figure 2(d). Nevertheless, the system and its order are identified correctly. This points out that there are linearly dependent parameters that are being identified. This is also indicated by the large variances computed.

To study the effect of truncation error when the pulse responses have not decayed to zero in the allowed window width, the pole radius in the complex case is increased to 0.9 while maintaining the same window width $p = 40$. Results in figure 3(a) show the parameter values drifting, while the variance is relatively small. The correct system order is used in this example. The prediction error is large, and the realization procedure identifies a fourth-order system. The identified pulse response, the reconstructed output, and the frequency response functions are significantly different from those of the actual system. The situation can be easily corrected by increasing the window width to $p = 80$ to reduce the truncation error. This is verified by the results presented in figure 3(b).

To eliminate the truncation error, the observer poles can all be placed at the origin. This is known as the deadbeat case because the observer pulse responses will go to zero in exactly a finite number of time steps. No estimate of the window width is needed, because once an assumption about the system order is made, the window width is automatically fixed. Results for the deadbeat case assuming the correct order are shown in figure 4(a). These results are similar to the real and complex cases, although the identified parameters are different. In all the cases discussed, the same input-output time histories are used for identification. Figures 4(b) and 4(c) show the deadbeat case when the assumed order is two and four, respectively. Figure 4(d) depicts results for order over-specification.

As with any numerical method, proper conditioning of the data is important. When identifying systems where the magnitudes of the input and output values are orders of magnitude apart, because of the use of different units for example, proper scaling of the numerical values is critical. This is true even for simple systems. The results shown here are scaled such that the input and output values have comparable magnitudes prior to application of the algorithm.

Multiple-Input Multiple-Output Examples

Identification of MIMO systems proceeds similarly to the SISO case. The model is the same, but now two inputs and two outputs are used for identification. The system is now of order 10. As in the SISO case, 200 data points are used in the identification algorithm.

First, consider the case where all prescribed observer poles are real. Initially the assumed order is set to $n = 4$ with corresponding pole locations at ± 0.2 and ± 0.3, and the window width p is set to 40. The top row of figure 5(a) shows the parameter convergence histories and the variance distribution for the least-squares solution. The parameters seem to have reached constant values, but some variations are still observed. Since the true order is 10, this is a case where the assumed order is less than the true order. This lack of freedom in the identified parameters prevents the prediction error from converging to zero, as shown in the second row of figure 5(a). Counting the number of nonzero singular values, the identified system order is found to be eight. The system pulse response, the reconstructed output, and the frequency response functions are in error. Figure 5(b) shows the results when the assumed order is increased to six. The parameters converge to constant values but the variances are large. Large variances indicate that some redundancy in the identified parameters has occurred. However, this does not affect the final answers. The prediction error converges to zero, and the system order is correctly identified to be 10. This is in contrast with the SISO case, where the identified system order does not exceed the assumed system order. It is important to note that for a given set of poles, the pole placement problem generally contains an infinite number of solutions for a multiple-output system. This results in additional freedom in the algorithm that is not present in the SISO case. It is this freedom that allows the identification of a system with a higher dimension than initially sought. One interesting aspect of the MIMO case is that when the observer pulse responses are realized to verify the prescribed pole locations, the apparent observer order is equal to the assumed order times the number of outputs. The resulting observer poles are those prescribed initially, but they are repeated as many times as the number of system outputs. A comparison of the pulse response, the reconstructed response, and the frequency response functions for the second output shows excellent agreement. Results for the first output are similar and not shown here.

Figure 6(a) shows results when the prescribed observer poles are complex. The poles are distributed evenly in the complex plane on a circle with radius $r = 0.5$. The assumed order is four. As in the real case, the window width p is set to 40. Results are similar to those of the real case in figure 5(a). Figure 6(b) shows the complex case when the assumed order is set to six. For the deadbeat algorithm, figures 7(a) and 7(b) show the identification results with assumed orders of four and six, respectively, when all prescribed poles are placed at the origin. Performance of the identification algorithm is similar to the previously discussed examples. As in the SISO case, if the assumed order is higher than the true order, the system can still be correctly identified, and the algorithm returns an identified model of minimal order.

Concluding Remarks

This paper formulates an algorithm for identification of linear multivariable systems from general input-output data. Data from either single or multiple sets of experiments can be used to identify or update the system model. For each data set, the initial condition may be arbitrary and need not be known. The procedure identifies the Markov parameters of an observer system instead of those of the actual system. The actual system Markov parameters are recovered from the observer Markov parameters and then used to realize a minimal state space model of the system. The embedded eigenvalue assignment procedure is used to specify the observer with asymptotically stable poles. The prescribed poles may be real, complex, or mixed real and complex. When all the prescribed poles are placed at the origin, this results in an identification algorithm with a deadbeat observer. Expressed in linear form, the observer Markov parameters can be solved for in one step for off-line computation, or recursively for on-line computation. The standard least-squares algorithm, which is used in one step of this identification procedure, may be replaced by other recursive parameter estimation algorithms. Identification procedures for both single-input single-output and multiple-input multiple-output systems are formulated, and numerical examples using noise-free simulated data are presented to illustrate the basic characteristics of the developed method.

Appendix A

SISO Mixed Real and Complex Eigenvalue Assignment

The identification procedure with mixed real and complex eigenvalue assignment can be derived in the same way as the case with real or complex eigenvalues. Among n prescribed eigenvalues, let n_r denote the number of prescribed real eigenvalues λ_i $(i = 1,\ 2,\ \ldots,\ n_r)$ and n_c the number of prescribed complex eigenvalues, $\lambda_i = \sigma_i \pm j\omega_i$ $(i = 1,\ 2,\ \ldots,\ n_c/2)$. Then write $\overline{A} = T^{-1}\Lambda_m T$, and

$$\Lambda_m = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_{n_r} & & & \\ & & & \begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} \sigma_{n_c/2} & \omega_{n_c/2} \\ -\omega_{n_c/2} & \sigma_{n_c/2} \end{bmatrix} \end{bmatrix} \tag{A1}$$

The product $(c\overline{A}^{\tau}b')$ in equation (24) becomes

$$\begin{aligned} c\overline{A}^{\tau}b' &= (c^*\Lambda_m^{\tau}b^*) \\ &= \lambda_1^{\tau}c_1^*b_1^* + \lambda_2^{\tau}c_2^*b_2^* + \ldots + \lambda_{n_r}^{\tau}c_{n_r}^*b_{n_r}^* + \sigma_1^{(\tau)}\left(c_{n_r+1}^*b_{n_r+1}^* + c_{n_r+2}^*b_{n_r+2}^*\right) \\ &\quad + \omega_1^{(\tau)}\left(c_{n_r+1}^*b_{n_r+2}^* - c_{n_r+2}^*b_{n_r+1}^*\right) + \ldots + \sigma_{n_c/2}^{(\tau)}\left(c_{n-1}^*b_{n-1}^* + c_n^*b_n^*\right) + \omega_{n_c/2}^{(\tau)}\left(c_{n-1}^*b_n^* - c_n^*b_{n-1}^*\right) \\ &= \alpha_m^T\lambda_m^{(\tau)} \end{aligned} \tag{A2}$$

Similarly,

$$\begin{aligned} c\overline{A}^{\tau}m &= (c^*\Lambda_m^{\tau}m^*) \\ &= \lambda_1^{\tau}c_1^*m_1^* + \lambda_2^{\tau}c_2^*m_2^* + \ldots + \lambda_{n_r}^{\tau}c_{n_r}^*m_{n_r}^* + \sigma_1^{(\tau)}\left(c_{n_r+1}^*m_{n_r+1}^* + c_{n_r+2}^*m_{n_r+2}^*\right) \\ &\quad + \omega_1^{(\tau)}\left(c_{n_r+1}^*m_{n_r+2}^* - c_{n_r+2}^*m_{n_r+1}^*\right) + \cdots + \sigma_{n_c/2}^{(\tau)}\left(c_{n-1}^*m_{n-1}^* + c_n^*m_n^*\right) + \omega_{n_c/2}^{(\tau)}\left(c_{n-1}^*m_n^* - c_n^*m_{n-1}^*\right) \\ &= -\beta_m^T\lambda_m^{(\tau)} \end{aligned} \tag{A3}$$

where

$$\alpha_m^T = \begin{bmatrix} \alpha^T & \alpha_c^T \end{bmatrix} \qquad \beta_m^T = \begin{bmatrix} \beta^T & \beta_c^T \end{bmatrix} \qquad \lambda_m^{(\tau)} = \begin{bmatrix} \lambda^{(\tau)^T} & \lambda_c^{(\tau)^T} \end{bmatrix}^T \tag{A4}$$

and

$$
\left.
\begin{aligned}
\alpha^T &= \begin{bmatrix} c_1^* b_1^* & c_2^* b_2^* & \cdots & c_{n_r}^* b_{n_r}^* \end{bmatrix} \\
\alpha_c^T &= \begin{bmatrix} \left(c_{n_r+1}^* b_{n_r+1}^* + c_{n_r+2}^* b_{n_r+2}^*\right) & \left(c_{n_r+1}^* b_{n_r+2}^* - c_{n_r+2}^* b_{n_r+1}^*\right) & \cdots & \left(c_{n-1}^* b_n^* - c_n^* b_{n-1}^*\right) \end{bmatrix} \\
\beta^T &= -\begin{bmatrix} c_1^* m_1^* & c_2^* m_2^* & \cdots & c_{n_r}^* m_{n_r}^* \end{bmatrix} \\
\beta_c^T &= -\begin{bmatrix} \left(c_{n_r+1}^* m_{n_r+1}^* + c_{n_r+2}^* m_{n_r+2}^*\right) & \left(c_{n_r+1}^* m_{n_r+2}^* - c_{n_r+2}^* m_{n_r+1}^*\right) & \cdots & \left(c_{n-1}^* m_n^* - c_n^* m_{n-1}^*\right) \end{bmatrix} \\
\lambda^{(\tau)} &= \begin{bmatrix} \lambda_1^\tau & \lambda_2^\tau & \cdots & \lambda_{n_r}^\tau \end{bmatrix} \\
\lambda_c^{(\tau)} &= \begin{bmatrix} \sigma_1^{(\tau)} & \omega_1^{(\tau)} & \sigma_2^{(\tau)} & \omega_2^{(\tau)} & \cdots & \sigma_{n_c/2}^{(\tau)} & \omega_{n_c/2}^{(\tau)} \end{bmatrix}^T \qquad \left(\sigma_i^{(0)} = 1, \quad \omega_i^{(0)} = 0\right)
\end{aligned}
\right\} \tag{A5}
$$

The elements c_i^*, b_i^*, and m_i^* ($i = 1, 2, \ldots, n$) are defined exactly the same way as in equations (28). Equation (24) now becomes

$$
\begin{aligned}
y(i) &= \alpha_m^T \sum_{\tau=0}^{p-1} \lambda_m^{(\tau)} u(i-\tau-1) + \beta_m^T \sum_{\tau=0}^{p-1} \lambda_m^{(\tau)} y(i-\tau-1) + du(i) \\
&= \gamma_m^T, {}_m(i-1)
\end{aligned} \tag{A6}
$$

which is again in linear form with

$$
\gamma_m^T = \begin{bmatrix} \alpha_m^T & \beta_m^T & d \end{bmatrix} \qquad , {}_m(i-1) = \begin{bmatrix} \phi_m(i-1) \\ \varphi_m(i-1) \\ u(i) \end{bmatrix} \tag{A7}
$$

where

$$
\left.
\begin{aligned}
\phi_m(i-1) &= \sum_{\tau=0}^{p-1} \lambda_m^{(\tau)} u(i-\tau-1) = \Im_m \underline{u}(i-p) \\
\varphi_m(i-1) &= \sum_{\tau=0}^{p-1} \lambda_m^{(\tau)} y(i-\tau-1) = \Im_m \underline{y}(i-p)
\end{aligned}
\right\} \tag{A8}
$$

The matrix $\Im_m$ is a Vandermonde-like matrix of n prescribed real and complex eigenvalues of magnitudes less than unity:

$$
\Im_m = \begin{bmatrix} \Im \\ \Im_c \end{bmatrix} \tag{A9}
$$

where

$$
\Im = \begin{bmatrix} \lambda_1^{p-1} & \lambda_1^{p-2} & \cdots & \lambda_1 & 1 \\ \lambda_2^{p-1} & \lambda_2^{p-2} & \cdots & \lambda_2 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{n_r}^{p-1} & \lambda_{n_r}^{p-2} & \cdots & \lambda_{n_r} & 1 \end{bmatrix} \qquad \Im_c = \begin{bmatrix} \sigma_1^{(p-1)} & \sigma_1^{(p-2)} & \cdots & \cdots & \sigma_1 & 1 \\ \omega_1^{(p-1)} & \omega_1^{(p-2)} & \cdots & \cdots & \omega_1 & 0 \\ \sigma_2^{(p-1)} & \sigma_2^{(p-2)} & \cdots & \cdots & \sigma_2 & 1 \\ \omega_2^{(p-1)} & \omega_2^{(p-2)} & \cdots & \cdots & \omega_2 & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \sigma_{n_c/2}^{(p-1)} & \sigma_{n_c/2}^{(p-2)} & \cdots & \cdots & \sigma_{n_c/2} & 1 \\ \omega_{n_c/2}^{(p-1)} & \omega_{n_c/2}^{(p-2)} & \cdots & \cdots & \omega_{n_c/2} & 0 \end{bmatrix} \tag{A10}
$$

and the $p \times 1$ input and output history vectors $\underline{u}(i-p)$ and $\underline{y}(i-p)$ are defined in equations (38). The standard recursive least-squares solution for the mixed real and complex eigenvalue case is obtained by simply replacing the estimated parameter vector $\widehat{\gamma}(i)$ by $\widehat{\gamma}_m(i)$, and , $(i-1)$ by , $_m(i-1)$ in equations (38). The observer Markov parameters $\overline{Y}_\tau$ ($\tau = 0,\ 1,\ 2,\ \cdots$) can be reconstructed according to

$$
\overline{Y}_\tau = \begin{bmatrix} \alpha_m^T \lambda_m^{(\tau)} & \beta_m^T \lambda_m^{(\tau)} \end{bmatrix} \equiv \begin{bmatrix} \overline{Y}_\tau^{(1)} & \overline{Y}_\tau^{(2)} \end{bmatrix} \tag{A11}
$$

Finally, the actual system Markov parameters can then be recovered as

$$
Y_\tau = \alpha_m^T \lambda^{(\tau)} + \beta_m^T \left(\sum_{i=0}^{\tau-1} \lambda_m^{(\tau)} Y_{\tau-i-1} + \lambda_m^{(\tau)} d \right) \tag{A12}
$$

Appendix B

Generalization to Multiple-Input Multiple-Output Systems

The developed identification theory for single-input single-output systems can be extended to the multivariable case. Consider the multivariable system in equations (1). The input-output relation in terms of the Markov parameters of an observer system is given in equation (9), which can be rewritten in ARMA model form as

$$y(i) = \sum_{\tau=0}^{p-1} \left(C\overline{A}^{\tau}B'\right) u(i-\tau-1) - \sum_{\tau=0}^{p-1} \left(C\overline{A}^{\tau}M\right) y(i-\tau-1) + Du(i) \tag{B1}$$

where

$$\overline{Y}_{\tau} = C\overline{A}^{\tau}\,\overline{B} = \begin{bmatrix} C\overline{A}^{\tau}B' & -C\overline{A}^{\tau}M \end{bmatrix} \qquad B' = B + MD$$

A recursive algorithm that computes the matrix coefficients of the ARMA model, and at the same time places the eigenvalues of $\overline{A}$ at prescribed locations, is derived in the following sections. These eigenvalues again may be real, complex, a combination of both, or zero (deadbeat).

MIMO Real Eigenvalue Assignment

Let the prescribed eigenvalues of $\overline{A} = T^{-1}\Lambda T$ be denoted by λ_i $(i = 1,\ 2,\ \cdots,\ n)$. Then the products $C\overline{A}^{\tau}B'$ and $C\overline{A}^{\tau}M$ become

$$C\overline{A}^{\tau}B' = CT^{-1}\Lambda^{\tau}TB' \qquad C\overline{A}^{\tau}M = CT^{-1}\Lambda^{\tau}TM \tag{B2}$$

If the elements of $C^* \equiv CT^{-1}$, $B^* \equiv TB'$, and $M^* \equiv TM$ are written explicitly as

$$C^* = \begin{bmatrix} c^*_{(1)} & c^*_{(2)} & \cdots & c^*_{(n)} \end{bmatrix} \qquad B^* = \begin{bmatrix} b^{*T}_{(1)} \\ b^{*T}_{(2)} \\ \vdots \\ b^{*T}_{(n)} \end{bmatrix} \qquad M^* = \begin{bmatrix} m^{*T}_{(1)} \\ m^{*T}_{(2)} \\ \vdots \\ m^{*T}_{(n)} \end{bmatrix} \tag{B3}$$

where $c^*_{(i)}$ denotes the ith column vector of the matrix C^*, and $b^{*T}_{(i)}$ and $m^{*T}_{(i)}$ $(i = 1,\ 2,\ \cdots,\ n)$ denote the ith row vectors of the matrices B^* and M^*, respectively, then the products in equation (B1) may be expressed as

$$C\overline{A}^{\tau}B' = C^*\Lambda^{\tau}B^* = \begin{bmatrix} c^*_{(1)}b^{*T}_{(1)} & c^*_{(2)}b^{*T}_{(2)} & \cdots & c^*_{(n)}b^{*T}_{(n)} \end{bmatrix} \begin{bmatrix} \underline{\lambda}^{(\tau)}_{1,m} \\ \underline{\lambda}^{(\tau)}_{2,m} \\ \vdots \\ \underline{\lambda}^{(\tau)}_{n,m} \end{bmatrix} \tag{B4}$$

$$C\overline{A}^{\,\tau}M = C^*\Lambda^\tau M^* = \begin{bmatrix} -c^*_{(1)}m^{*T}_{(1)} & -c^*_{(2)}m^{*T}_{(2)} & \cdots & -c^*_{(n)}m^{*T}_{(n)} \end{bmatrix} \begin{bmatrix} \underline{\lambda}^{(\tau)}_{1,q} \\ \underline{\lambda}^{(\tau)}_{2,q} \\ \vdots \\ \underline{\lambda}^{(\tau)}_{n,q} \end{bmatrix} \tag{B5}$$

where $\underline{\lambda}^{(\tau)}_{i,m}$ and $\underline{\lambda}^{(\tau)}_{i,q}$ are $m \times m$ and $q \times q$ diagonal matrices of the eigenvalue λ_i repeated m and q times, respectively, i.e.,

$$\underline{\lambda}^{(\tau)}_{i,m} = \begin{bmatrix} \lambda_i^\tau & & \\ & \ddots & \\ & & \lambda_i^\tau \end{bmatrix}_{m \times m} \qquad \underline{\lambda}^{(\tau)}_{i,q} = \begin{bmatrix} \lambda_i^\tau & & \\ & \ddots & \\ & & \lambda_i^\tau \end{bmatrix}_{q \times q} \tag{B6}$$

With the following simplifying definitions as in equations (31)

$$\left.\begin{aligned} \underline{\alpha} &= \begin{bmatrix} c^*_{(1)}b^{*T}_{(1)} & c^*_{(2)}b^{*T}_{(2)} & \cdots & c^*_{(n)}b^{*T}_{(n)} \end{bmatrix} \\ \underline{\beta} &= \begin{bmatrix} -c^*_{(1)}m^{*T}_{(1)} & -c^*_{(2)}m^{*T}_{(2)} & \cdots & -c^*_{(n)}m^{*T}_{(n)} \end{bmatrix} \\ \underline{\lambda}^{(\tau)}_{m} &= \begin{bmatrix} \underline{\lambda}^{(\tau)}_{1,m} & \underline{\lambda}^{(\tau)}_{2,m} & \cdots & \underline{\lambda}^{(\tau)}_{n,m} \end{bmatrix}^T \\ \underline{\lambda}^{(\tau)}_{q} &= \begin{bmatrix} \underline{\lambda}^{(\tau)}_{1,q} & \underline{\lambda}^{(\tau)}_{2,q} & \cdots & \underline{\lambda}^{(\tau)}_{n,q} \end{bmatrix}^T \end{aligned}\right\} \tag{B7}$$

equation (B1) becomes

$$\begin{aligned} y(i) &= \underline{\alpha}\sum_{\tau=0}^{p-1}\underline{\lambda}^{(\tau)}_{m}u(i-\tau-1) + \underline{\beta}\sum_{\tau=0}^{p-1}\underline{\lambda}^{(\tau)}_{q}y(i-\tau-1) + Du(i) \\ &= \underline{\gamma}\,\underline{,}(i-1) \end{aligned} \tag{B8}$$

where $u(i)$ and $y(i)$ are $m \times 1$ and $q \times 1$ input and output vectors, respectively. The above equation is in a linear form with the unknown parameters in the matrices $\underline{\alpha}$, $\underline{\beta}$, and D with

$$\underline{\gamma} = \begin{bmatrix} \underline{\alpha} & \underline{\beta} & D \end{bmatrix} \qquad \underline{,}(i-1) = \begin{bmatrix} \underline{\phi}(i-1) \\ \underline{\varphi}(i-1) \\ u(i) \end{bmatrix} \tag{B9}$$

where

$$\left.\begin{aligned} \underline{\phi}(i-1) &= \sum_{\tau=0}^{p-1} \underline{\lambda}_m^{(\tau)} u(i-\tau-1) = \underline{\Im}_m \underline{u}(i-p) \\ \underline{\varphi}(i-1) &= \sum_{\tau=0}^{p-1} \underline{\lambda}_q^{(\tau)} y(i-\tau-1) = \underline{\Im}_q \underline{y}(i-p) \end{aligned}\right\} \tag{B10}$$

and

$$\begin{aligned} \underline{\Im}_m &= \begin{bmatrix} \underline{\lambda}_m^{(p-1)} & \underline{\lambda}_m^{(p-2)} & \cdots & \underline{\lambda}_m^{(1)} & \underline{\lambda}_m^{(0)} \end{bmatrix} \\ &= \begin{bmatrix} \underline{\lambda}_{1,m}^{(p-1)} & \underline{\lambda}_{1,m}^{(p-2)} & \cdots & \underline{\lambda}_{1,m}^{(1)} & I_{m\times m} \\ \underline{\lambda}_{2,m}^{(p-1)} & \underline{\lambda}_{2,m}^{(p-2)} & \cdots & \underline{\lambda}_{2,m}^{(1)} & I_{m\times m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \underline{\lambda}_{n,m}^{(p-1)} & \underline{\lambda}_{n,m}^{(p-2)} & \cdots & \underline{\lambda}_{n,m}^{(1)} & I_{m\times m} \end{bmatrix} \end{aligned} \tag{B11}$$

In terms of the prescribed eigenvalues, $\underline{\Im}_m$ has the following structure:

$$\underline{\Im}_m = \begin{bmatrix} \begin{bmatrix} \lambda_1^{p-1} & & \\ & \ddots & \\ & & \lambda_1^{p-1} \end{bmatrix} & \begin{bmatrix} \lambda_1^{p-2} & & \\ & \ddots & \\ & & \lambda_1^{p-2} \end{bmatrix} & \cdots & \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{bmatrix} & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{m\times m} \\ \begin{bmatrix} \lambda_2^{p-1} & & \\ & \ddots & \\ & & \lambda_2^{p-1} \end{bmatrix} & \begin{bmatrix} \lambda_2^{p-2} & & \\ & \ddots & \\ & & \lambda_2^{p-2} \end{bmatrix} & \cdots & \begin{bmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{bmatrix} & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{m\times m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \begin{bmatrix} \lambda_n^{p-1} & & \\ & \ddots & \\ & & \lambda_n^{p-1} \end{bmatrix} & \begin{bmatrix} \lambda_n^{p-2} & & \\ & \ddots & \\ & & \lambda_n^{p-2} \end{bmatrix} & \cdots & \begin{bmatrix} \lambda_n & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{m\times m} \end{bmatrix}$$

Similarly,

$$\underline{\Im}_q = \begin{bmatrix} \underline{\lambda}_q^{(p-1)} & \underline{\lambda}_q^{(p-2)} & \cdots & \underline{\lambda}_q^{(1)} & \underline{\lambda}_q^{(0)} \end{bmatrix} \tag{B12}$$

which has the same general structure as $\underline{\Im}_m$ except the block matrices are of dimensions $q \times q$. The $mp \times 1$ input history vector $\underline{u}(i-p)$ and the $qp \times 1$ output history vector $\underline{y}(i-p)$ are defined the same way as in equations (37) except that the input $u(i)$ is of dimensions $m \times 1$, and the output $y(i)$ is of dimensions $q \times 1$. Equation (B8) is in linear form; the parameter matrix $\underline{\gamma}$ can be solved for directly from input-output data. For on-line computation, the recursive least-squares solution to the parameter matrix $\underline{\gamma}$ is given as

$$\left.\begin{aligned}
\underline{\widehat{\gamma}}^T(i) &= \underline{\widehat{\gamma}}^T(i-1) + \frac{\Re(i-2)\underline{,}(i-1)}{1+\underline{,}(i-1)^T\Re(i-2)\underline{,}(i-1)}\left[y(i)-\underline{\widehat{\gamma}}(i-1)\underline{,}(i-1)\right]^T \\
\Re(i-1) &= \Re(i-2) - \frac{\Re(i-2)\underline{,}(i-1)\underline{,}(i-1)^T\Re(i-2)}{1+\underline{,}(i-1)^T\Re(i-2)\underline{,}(i-1)}
\end{aligned}\right\} \tag{B13}$$

The observer Markov parameters $\overline{Y}_\tau$ ($\tau = 0,\ 1,\ 2,\ \cdots$) can be reconstructed according to

$$\begin{aligned}
\overline{Y}_\tau &= C\overline{A}^\tau\overline{B} = C\overline{A}^\tau\begin{bmatrix} B' & -M \end{bmatrix} \\
&= \begin{bmatrix} \underline{\alpha}\underline{\lambda}_m^{(\tau)} & \underline{\beta}\underline{\lambda}_q^{(\tau)} \end{bmatrix} = \begin{bmatrix} \overline{Y}_\tau^{(1)} & \overline{Y}_\tau^{(2)} \end{bmatrix}
\end{aligned} \tag{B14}$$

Finally, the actual system Markov parameters can then be recovered from the reconstructed observer Markov parameters according to equation (20) as

$$\begin{aligned}
Y_\tau &= \overline{Y}_\tau^{(1)} + \sum_{i=0}^{\tau-1}\overline{Y}_\tau^{(2)}Y_{\tau-i-1} + \overline{Y}_\tau^{(2)}D \\
&= \underline{\alpha}\underline{\lambda}_m^{(\tau)} + \underline{\beta}\left(\sum_{i=0}^{\tau-1}\underline{\lambda}_q^{(\tau)}Y_{\tau-i-1} + \underline{\lambda}_q^{(\tau)}D\right)
\end{aligned} \tag{B15}$$

MIMO Complex Eigenvalue Assignment

The complex eigenvalue assignment for the multiple-input multiple-output case can be derived by setting $\overline{A} = T^{-1}\Lambda_c T$, where Λ_c is given as in equation (41). The prescribed complex conjugate pairs of eigenvalues are denoted $\lambda_i = \sigma_i \pm j\omega_i$ ($i = 1,\ 2,\ \ldots,\ n/2$). Using the same notation for vectors formed by the columns and rows of C^* and B^*, respectively, the products in equation (B1) may be expressed as

$$\begin{aligned}
C\overline{A}^\tau B' &= C^*\Lambda_c^\tau B^* \\
&= \sigma_1^{(\tau)}\left(c_{(1)}^*b_{(1)}^{*T} + c_{(2)}^*b_{(2)}^{*T}\right) + \omega_1^{(\tau)}\left(c_{(1)}^*b_{(2)}^{*T} - c_{(2)}^*b_{(1)}^{*T}\right) + \sigma_2^{(\tau)}\left(c_{(3)}^*b_{(3)}^{*T} + c_{(4)}^*b_{(4)}^{*T}\right) \\
&\quad + \omega_2^{(\tau)}\left(c_{(3)}^*b_{(4)}^{*T} - c_{(4)}^*b_{(3)}^{*T}\right) + \cdots + \sigma_{n/2}^{(\tau)}\left(c_{(n-1)}^*b_{(n-1)}^{*T} + c_{(n)}^*b_{(n)}^{*T}\right) + \omega_{n/2}^{(\tau)}\left(c_{(n-1)}^*b_{(n)}^{*T} - c_{(n)}^*b_{(n-1)}^{*T}\right) \\
&= \underline{\alpha_c}\underline{\lambda}_{c,m}^{(\tau)} \\
C\overline{A}^\tau M &= C^*\Lambda_c^\tau M^* \\
&= \sigma_1^{(\tau)}\left(c_{(1)}^*m_{(1)}^{*T} + c_{(2)}^*m_{(2)}^{*T}\right) + \omega_1^{(\tau)}\left(c_{(1)}^*m_{(2)}^{*T} - c_{(2)}^*m_{(1)}^{*T}\right) + \sigma_2^{(\tau)}\left(c_{(3)}^*m_{(3)}^{*T} + c_{(4)}^*m_{(4)}^{*T}\right) \\
&\quad + \omega_2^{(\tau)}\left(c_{(3)}^*m_{(4)}^{*T} - c_{(4)}^*m_{(3)}^{*T}\right) + \cdots + \sigma_{n/2}^{(\tau)}\left(c_{(n-1)}^*m_{(n-1)}^{*T} + c_{(n)}^*m_{(n)}^{*T}\right) + \omega_{n/2}^{(\tau)}\left(c_{(n-1)}^*m_{(n)}^{*T} - c_{(n)}^*m_{(n-1)}^{*T}\right) \\
&= \underline{\beta_c}\underline{\lambda}_{c,q}^{(\tau)}
\end{aligned} \tag{B16}$$

where

$$\underline{\alpha_c} = \left[c^*_{(1)} b^{*T}_{(1)} + c^*_{(2)} b^{*T}_{(2)} \quad c^*_{(1)} b^{*T}_{(2)} - c^*_{(2)} b^{*T}_{(1)} \quad \cdots \quad c^*_{(n-1)} b^{*T}_{(n-1)} + c^*_{(n)} b^{*T}_{(n)} \quad c^*_{(n-1)} b^{*T}_{(n)} - c^*_{(n)} b^{*T}_{(n-1)} \right]$$

$$\underline{\beta_c} = -\left[c^*_{(1)} m^{*T}_{(1)} + c^*_{(2)} m^{*T}_{(2)} \quad c^*_{(1)} m^{*T}_{(2)} - c^*_{(2)} m^{*T}_{(1)} \quad \cdots \quad c^*_{(n-1)} m^{*T}_{(n-1)} + c^*_{(n)} m^{*T}_{(n)} \quad c^*_{(n-1)} m^{*T}_{(n)} - c^*_{(n)} m^{*T}_{(n-1)} \right]$$

$$\underline{\lambda}^{(\tau)}_{c,m} = \left[\underline{\sigma}^{(\tau)}_{1,m} \quad \underline{\omega}^{(\tau)}_{1,m} \quad \underline{\sigma}^{(\tau)}_{2,m} \quad \underline{\omega}^{(\tau)}_{2,m} \quad \cdots \quad \underline{\sigma}^{(\tau)}_{n/2,m} \quad \underline{\omega}^{(\tau)}_{n/2,m} \right]^T \qquad \left(\underline{\sigma}^{(0)}_{i,m} = I_{m\times m}, \quad \underline{\omega}^{(0)}_{i,m} = 0_{m\times m} \right)$$

$$\underline{\lambda}^{(\tau)}_{c,q} = \left[\underline{\sigma}^{(\tau)}_{1,q} \quad \underline{\omega}^{(\tau)}_{1,q} \quad \underline{\sigma}^{(\tau)}_{2,q} \quad \underline{\omega}^{(\tau)}_{2,q} \quad \cdots \quad \underline{\sigma}^{(\tau)}_{n/2,q} \quad \underline{\omega}^{(\tau)}_{n/2,q} \right]^T \qquad \left(\underline{\sigma}^{(0)}_{i,q} = I_{q\times q}, \quad \underline{\omega}^{(0)}_{i,q} = 0_{q\times q} \right)$$

The matrices $\underline{\sigma}^{(\tau)}_{i,m}$ and $\underline{\omega}^{(\tau)}_{i,m}$ are $m \times m$ diagonal matrices formed by $\sigma^{(\tau)}_i$ and $\omega^{(\tau)}_i$ repeated m times, respectively; $\sigma^{(\tau)}_i$ and $\omega^{(\tau)}_i$ are the elements associated with the complex eigenvalue pair $\lambda_i = \sigma_i \pm j\omega_i$ $(i = 1,\ 2,\ \cdots,\ n/2)$ as defined in equation (42), i.e.,

$$\underline{\sigma}^{(\tau)}_{i,m} = \begin{bmatrix} \sigma^{(\tau)}_i & & \\ & \ddots & \\ & & \sigma^{(\tau)}_i \end{bmatrix}_{m\times m} \qquad \underline{\omega}^{(\tau)}_{i,m} = \begin{bmatrix} \omega^{(\tau)}_i & & \\ & \ddots & \\ & & \omega^{(\tau)}_i \end{bmatrix}_{m\times m} \tag{B17}$$

Similar definitions apply for $\underline{\sigma}^{(\tau)}_{i,q}$ and $\underline{\omega}^{(\tau)}_{i,q}$ simply by replacing m by q. Equation (B1) now becomes

$$\begin{aligned} y(i) &= \underline{\alpha_c} \sum_{\tau=0}^{p-1} \underline{\lambda}^{(\tau)}_{c,m} u(i-\tau-1) + \underline{\beta_c} \sum_{\tau=0}^{p-1} \underline{\lambda}^{(\tau)}_{c,q} y(i-\tau-1) + Du(i) \\ &= \underline{\gamma_c\ ,\ _c}(i-1) \end{aligned} \tag{B18}$$

where

$$\underline{\gamma_c} = \begin{bmatrix} \underline{\alpha_c} & \underline{\beta_c} & D \end{bmatrix} \qquad \underline{,\ _c}(i-1) = \begin{bmatrix} \underline{\phi_c}(i-1) \\ \underline{\varphi_c}(i-1) \\ u(i) \end{bmatrix} \tag{B19}$$

The vectors $\underline{\phi_c}(i-1)$ and $\underline{\varphi_c}(i-1)$ in equations (B19) are given as

$$\left.\begin{aligned} \underline{\phi_c}(i-1) &= \sum_{\tau=0}^{p-1} \underline{\lambda}^{(\tau)}_{c,m} u(i-\tau-1) = \underline{\Im_{c,m}}\, \underline{u}(i-p) \\ \underline{\varphi_c}(i-1) &= \sum_{\tau=0}^{p-1} \underline{\lambda}^{(\tau)}_{c,q} y(i-\tau-1) = \underline{\Im_{c,q}}\, \underline{y}(i-p) \end{aligned}\right\} \tag{B20}$$

and

$$\underline{\Im}_{c,m} = \begin{bmatrix} \underline{\lambda}_{c,m}^{(p-1)} & \underline{\lambda}_{c,m}^{(p-2)} & \cdots & \underline{\lambda}_{c,m}^{(1)} & \underline{\lambda}_{c,m}^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\sigma}_{1,m}^{(p-1)} & \underline{\sigma}_{1,m}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{1,m}^{(1)} & I_{m\times m} \\ \underline{\omega}_{1,m}^{(p-1)} & \underline{\omega}_{1,m}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{1,m}^{(1)} & 0_{m\times m} \\ \underline{\sigma}_{2,m}^{(p-1)} & \underline{\sigma}_{2,m}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{2,m}^{(1)} & I_{m\times m} \\ \underline{\omega}_{2,m}^{(p-1)} & \underline{\omega}_{2,m}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{2,m}^{(1)} & 0_{m\times m} \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \underline{\sigma}_{n/2,m}^{(p-1)} & \underline{\sigma}_{n/2,m}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{n/2,m}^{(1)} & I_{m\times m} \\ \underline{\omega}_{n/2,m}^{(p-1)} & \underline{\omega}_{n/2,m}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{n/2,m}^{(1)} & 0_{m\times m} \end{bmatrix} \tag{B21}$$

and similarly,

$$\underline{\Im}_{c,q} = \begin{bmatrix} \underline{\lambda}_{c,q}^{(p-1)} & \underline{\lambda}_{c,q}^{(p-2)} & \cdots & \underline{\lambda}_{c,q}^{(1)} & \underline{\lambda}_{c,q}^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\sigma}_{1,q}^{(p-1)} & \underline{\sigma}_{1,q}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{1,q}^{(1)} & I_{q\times q} \\ \underline{\omega}_{1,q}^{(p-1)} & \underline{\omega}_{1,q}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{1,q}^{(1)} & 0_{q\times q} \\ \underline{\sigma}_{2,q}^{(p-1)} & \underline{\sigma}_{2,q}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{2,q}^{(1)} & I_{q\times q} \\ \underline{\omega}_{2,q}^{(p-1)} & \underline{\omega}_{2,q}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{2,q}^{(1)} & 0_{q\times q} \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \underline{\sigma}_{n/2,q}^{(p-1)} & \underline{\sigma}_{n/2,q}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{n/2,q}^{(1)} & I_{q\times q} \\ \underline{\omega}_{n/2,q}^{(p-1)} & \underline{\omega}_{n/2,q}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{n/2,q}^{(1)} & 0_{q\times q} \end{bmatrix} \tag{B22}$$

In terms of the prescribed complex eigenvalues, $\underline{\Im}_{c,m}$ has the following structure:

$$
\underline{\Im}_{c,m} = \begin{bmatrix}
\begin{bmatrix} \sigma_1^{(p-1)} & & \\ & \ddots & \\ & & \sigma_1^{(p-1)} \end{bmatrix} & \begin{bmatrix} \sigma_1^{(p-2)} & & \\ & \ddots & \\ & & \sigma_1^{(p-2)} \end{bmatrix} & \cdots\ \cdots & \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_1 \end{bmatrix} & \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}_{m \times m} \\
\begin{bmatrix} \omega_1^{(p-1)} & & \\ & \ddots & \\ & & \omega_1^{(p-1)} \end{bmatrix} & \begin{bmatrix} \omega_1^{(p-2)} & & \\ & \ddots & \\ & & \omega_1^{(p-2)} \end{bmatrix} & \cdots\ \cdots & \begin{bmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_1 \end{bmatrix} & \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}_{m \times m} \\
\begin{bmatrix} \sigma_2^{(p-1)} & & \\ & \ddots & \\ & & \sigma_2^{(p-1)} \end{bmatrix} & \begin{bmatrix} \sigma_2^{(p-2)} & & \\ & \ddots & \\ & & \sigma_2^{(p-2)} \end{bmatrix} & \cdots\ \cdots & \begin{bmatrix} \sigma_2 & & \\ & \ddots & \\ & & \sigma_2 \end{bmatrix} & \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}_{m \times m} \\
\begin{bmatrix} \omega_2^{(p-1)} & & \\ & \ddots & \\ & & \omega_2^{(p-1)} \end{bmatrix} & \begin{bmatrix} \omega_2^{(p-2)} & & \\ & \ddots & \\ & & \omega_2^{(p-2)} \end{bmatrix} & \cdots\ \cdots & \begin{bmatrix} \omega_2 & & \\ & \ddots & \\ & & \omega_2 \end{bmatrix} & \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}_{m \times m} \\
\vdots & \vdots & \cdots\ \cdots & \vdots & \vdots \\
\begin{bmatrix} \sigma_{n/2}^{(p-1)} & & \\ & \ddots & \\ & & \sigma_{n/2}^{(p-1)} \end{bmatrix} & \begin{bmatrix} \sigma_{n/2}^{(p-2)} & & \\ & \ddots & \\ & & \sigma_{n/2}^{(p-2)} \end{bmatrix} & \cdots\ \cdots & \begin{bmatrix} \sigma_{n/2} & & \\ & \ddots & \\ & & \sigma_{n/2} \end{bmatrix} & \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}_{m \times m} \\
\begin{bmatrix} \omega_{n/2}^{(p-1)} & & \\ & \ddots & \\ & & \omega_{n/2}^{(p-1)} \end{bmatrix} & \begin{bmatrix} \omega_{n/2}^{(p-2)} & & \\ & \ddots & \\ & & \omega_{n/2}^{(p-2)} \end{bmatrix} & \cdots\ \cdots & \begin{bmatrix} \omega_{n/2} & & \\ & \ddots & \\ & & \omega_{n/2} \end{bmatrix} & \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}_{m \times n}
\end{bmatrix}
$$

The structure for $\underline{\Im}_{c,q}$ is similar. The recursive solution to equation (B18) can be obtained by replacing $\underline{\widehat{\gamma}}$ by $\underline{\widehat{\gamma}_c}$, and $\underline{\Gamma}(i-1)$ by $\underline{\Gamma_c}(i-1)$ in equations (B13). The observer Markov parameters and the actual Markov parameters can then be computed as

$$
\begin{aligned}
\overline{Y}_\tau = C\overline{A}^\tau \overline{B} &= C\overline{A}^\tau \begin{bmatrix} B' & -M \end{bmatrix} \\
&= \begin{bmatrix} \underline{\alpha_c} \underline{\lambda}_{c,m}^{(\tau)} & \underline{\beta_c} \underline{\lambda}_{c,q}^{(\tau)} \end{bmatrix} = \begin{bmatrix} \overline{Y}_\tau^{(1)} & \overline{Y}_\tau^{(2)} \end{bmatrix}
\end{aligned} \tag{B23}
$$

$$
\begin{aligned}
Y_\tau &= \overline{Y}_\tau^{(1)} + \sum_{i=0}^{\tau-1} \overline{Y}_\tau^{(2)} Y_{\tau-i-1} + \overline{Y}_\tau^{(2)} D \\
&= \underline{\alpha}_c \underline{\lambda}_{c,m}^{(\tau)} + \underline{\beta_c} \left(\sum_{i=0}^{\tau-1} \underline{\lambda}_{c,q}^{(\tau)} Y_{\tau-i-1} + \underline{\lambda}_{c,q}^{(\tau)} D \right)
\end{aligned} \tag{B24}
$$

MIMO Mixed Real and Complex Eigenvalue Assignment

Among n prescribed eigenvalues, let n_r denote the number of prescribed real eigenvalues λ_i $(i = 1,\ 2,\ \ldots,\ n_r)$ and n_c the number of prescribed complex eigenvalues $\sigma_i \pm j\omega_i$ $(i = 1,\ 2,\ \ldots,\ n_c/2)$. Then write $\overline{A} = T^{-1}\Lambda_m T$, and Λ_m as in equation (A1), and define

$$
\underline{\alpha} = \begin{bmatrix} \underline{\alpha} & \underline{\alpha}_c \end{bmatrix} \qquad \underline{\beta_m} = \begin{bmatrix} \underline{\beta} & \underline{\beta_c} \end{bmatrix} \qquad \underline{\lambda}_{m,m}^{(\tau)} = \begin{bmatrix} \underline{\lambda}_m^{(\tau)T} & \underline{\lambda}_{c,m}^{(\tau)T} \end{bmatrix} \qquad \underline{\lambda}_{m,q}^{(\tau)} = \begin{bmatrix} \underline{\lambda}_q^{(\tau)T} & \underline{\lambda}_{c,q}^{(\tau)T} \end{bmatrix}^T \tag{B25}
$$

where

$$
\begin{aligned}
\underline{\alpha} &= \begin{bmatrix} c_{(1)}^* b_{(1)}^{*T} & c_{(2)}^* b_{(2)}^{*T} & \cdots & c_{(n_r)}^* b_{(n_r)}^{*T} \end{bmatrix} \\
\underline{\alpha}_c &= \begin{bmatrix} \left(c_{(n_r+1)}^* b_{(n_r+1)}^{*T} + c_{(n_r+2)}^* b_{(n_r+2)}^{*T} \right) & \cdots & \cdots & \cdots & \left(c_{(n-1)}^* b_{(n)}^{*T} - c_{(n)}^* b_{(n-1)}^{*T} \right) \end{bmatrix} \\
\underline{\beta} &= -\begin{bmatrix} c_{(1)}^* m_{(1)}^{*T} & c_{(2)}^* m_{(2)}^{*T} & \cdots & c_{(n_r)}^* m_{(n_r)}^{*T} \end{bmatrix} \\
\underline{\beta_c} &= -\begin{bmatrix} \left(c_{(n_r+1)}^* m_{(n_r+1)}^{*T} + c_{(n_r+2)}^* m_{(n_r+2)}^{*T} \right) & \cdots & \cdots & \cdots & \left(c_{(n-1)}^* m_{(n)}^{*T} - c_{(n)}^* m_{(n-1)}^{*T} \right) \end{bmatrix}
\end{aligned}
$$

and

$$
\left.
\begin{aligned}
\underline{\lambda}_m^{(\tau)} &= \begin{bmatrix} \underline{\lambda}_{1,m}^{(\tau)} & \underline{\lambda}_{2,m}^{(\tau)} & \cdots & \underline{\lambda}_{n_r,m}^{(\tau)} \end{bmatrix}^T \\
\underline{\lambda}_q^{(\tau)} &= \begin{bmatrix} \underline{\lambda}_{1,q}^{(\tau)} & \underline{\lambda}_{2,q}^{(\tau)} & \cdots & \underline{\lambda}_{n_r,q}^{(\tau)} \end{bmatrix}^T \\
\underline{\lambda}_{c,m}^{(\tau)} &= \begin{bmatrix} \underline{\sigma}_{1,m}^{(\tau)} & \underline{\omega}_{1,m}^{(\tau)} & \underline{\sigma}_{2,m}^{(\tau)} & \underline{\omega}_{2,m}^{(\tau)} & \cdots & \underline{\sigma}_{n_c/2,m}^{(\tau)} & \underline{\omega}_{n_c/2,m}^{(\tau)} \end{bmatrix}^T,\ \underline{\sigma}_{i,m}^{(0)} = I_{m\times m},\ \underline{\omega}_{i,m}^{(0)} = 0_{m\times m} \\
\underline{\lambda}_{c,q}^{(\tau)} &= \begin{bmatrix} \underline{\sigma}_{1,q}^{(\tau)} & \underline{\omega}_{1,q}^{(\tau)} & \underline{\sigma}_{2,q}^{(\tau)} & \underline{\omega}_{2,q}^{(\tau)} & \cdots & \underline{\sigma}_{n_c/2,q}^{(\tau)} & \underline{\omega}_{n_c/2,q}^{(\tau)} \end{bmatrix}^T,\ \underline{\sigma}_{i,q}^{(0)} = I_{q\times q},\ \underline{\omega}_{i,q}^{(0)} = 0_{q\times q}
\end{aligned}
\right\} \tag{B26}
$$

Equation (B1) may be expressed as

$$
\begin{aligned}
y(i) &= \underline{\alpha}_m \sum_{\tau=0}^{p-1} \underline{\lambda}_{m,m}^{(\tau)} u(i-\tau-1) + \underline{\beta_m} \sum_{\tau=0}^{p-1} \underline{\lambda}_{m,q}^{(\tau)} y(i-\tau-1) + Du(i) \\
&= \underline{\gamma_{m,\ m}}(i-1)
\end{aligned} \tag{B27}
$$

where

$$\underline{\gamma_m} = \begin{bmatrix} \underline{\alpha}_m & \underline{\beta}_m & D \end{bmatrix} \qquad \underline{,\ _m}(i-1) = \begin{bmatrix} \underline{\phi_m}(i-1) \\ \underline{\varphi_m}(i-1) \\ u(i) \end{bmatrix} \tag{B28}$$

$$\left.\begin{aligned} \underline{\phi_m}(i-1) &= \sum_{\tau=0}^{p-1} \underline{\lambda}_{m,m}^{(\tau)} u(i-\tau-1) = \underline{\Im}_{m,m}\underline{u}(i-p) \\ \underline{\varphi_m}(i-1) &= \sum_{\tau=0}^{p-1} \underline{\lambda}_{m,q}^{(\tau)} y(i-\tau-1) = \underline{\Im}_{m,q}\underline{y}(i-p) \end{aligned}\right\} \tag{B29}$$

The matrix $\underline{\Im}_{m,m}$ includes elements formed from both real and complex prescribed eigenvalues:

$$\underline{\Im}_{m,m} = \begin{bmatrix} \underline{\Im}_m \\ \underline{\Im}_{c,m} \end{bmatrix} \tag{B30}$$

where

$$\underline{\Im}_m = \begin{bmatrix} \underline{\lambda}_{1,m}^{(p-1)} & \underline{\lambda}_{1,m}^{(p-2)} & \cdots & \underline{\lambda}_{1,m}^{(1)} & I_{m\times m} \\ \underline{\lambda}_{2,m}^{(p-1)} & \underline{\lambda}_{2,m}^{(p-2)} & \cdots & \underline{\lambda}_{2,m}^{(1)} & I_{m\times m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \underline{\lambda}_{n_r,m}^{(p-1)} & \underline{\lambda}_{n_r,m}^{(p-2)} & \cdots & \underline{\lambda}_{n_r,m}^{(1)} & I_{m\times m} \end{bmatrix} \qquad \underline{\Im}_{c,m} = \begin{bmatrix} \underline{\sigma}_{1,m}^{(p-1)} & \underline{\sigma}_{1,m}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{1,m}^{(1)} & I_{m\times m} \\ \underline{\omega}_{1,m}^{(p-1)} & \underline{\omega}_{1,m}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{1,m}^{(1)} & 0_{m\times m} \\ \underline{\sigma}_{2,m}^{(p-1)} & \underline{\sigma}_{2,m}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{2,m}^{(1)} & I_{m\times m} \\ \underline{\omega}_{2,m}^{(p-1)} & \underline{\omega}_{2,m}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{2,m}^{(1)} & 0_{m\times m} \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \underline{\sigma}_{n_c/2,m}^{(p-1)} & \underline{\sigma}_{n_c/2,m}^{(p-2)} & \cdots & \cdots & \underline{\sigma}_{n_c/2,m}^{(1)} & I_{m\times m} \\ \underline{\omega}_{n_c/2,m}^{(p-1)} & \underline{\omega}_{n_c/2,m}^{(p-2)} & \cdots & \cdots & \underline{\omega}_{n_c/2,m}^{(1)} & 0_{m\times m} \end{bmatrix}$$

The diagonal matrices $\underline{\lambda}_{i,m}^{(\tau)}$, $\underline{\sigma}_{i,m}^{(\tau)}$, $\underline{\omega}_{i,m}^{(\tau)}$ in $\underline{\Im}_{m,m}$ are of dimensions $m \times m$, $i = 1,\ 2,\ \ldots,\ n_r$, or $n_c/2$, and $\tau = 1,\ 2,\ \ldots,\ p-1$. Similar structures apply for $\underline{\Im}_{m,q}$, which is composed of the matrices $\underline{\lambda}_{i,q}^{(\tau)}$, $\underline{\sigma}_{i,q}^{(\tau)}$, $\underline{\omega}_{i,q}^{(\tau)}$ of dimensions $q \times q$ instead. The recursive solution to the parameter matrix $\underline{\gamma_m}$ is obvious. The observer Markov parameters and the actual system Markov parameters are simply

$$\overline{Y}_\tau = \begin{bmatrix} \underline{\alpha}_m \underline{\lambda}_{m,m}^{(\tau)} & \underline{\beta_m} \underline{\lambda}_{m,q}^{(\tau)} \end{bmatrix} = \begin{bmatrix} \overline{Y}_\tau^{(1)} & \overline{Y}_\tau^{(2)} \end{bmatrix} \tag{B31}$$

$$Y_\tau = \underline{\alpha}_m \underline{\lambda}_{m,m}^{(\tau)} + \underline{\beta_m} \left(\sum_{i=0}^{\tau-1} \underline{\lambda}_{m,q}^{(\tau)} Y_{\tau-i-1} + \underline{\lambda}_{m,q}^{(\tau)} D \right) \tag{B32}$$

MIMO Deadbeat Eigenvalue Assignment

In the deadbeat case, all eigenvalues of the observer are placed at the origin. The corresponding Markov parameters will vanish identically after n time steps. In other words,

$\overline{Y}_\tau \equiv 0$ for $\tau = n,\ n+1,\ n+2,\ \ldots$. Let M_d denote the deadbeat observer gain for the multiple-input multiple-output case. The input-output description is given in terms of the observer Markov parameters as

$$
\begin{aligned}
y(i) &= \sum_{\tau=0}^{n-1} \left(C\overline{A}^{\tau} B'_d\right) u(i-\tau-1) - \sum_{\tau=0}^{n-1} \left(C\overline{A}^{\tau} M_d\right) y(i-\tau-1) + Du(i) \\
&= \sum_{\tau=0}^{n-1} \overline{Y}_{\tau}^{(1)} u(i-\tau-1) - \sum_{\tau=0}^{n-1} \overline{Y}_{\tau}^{(2)} y(i-\tau-1) + Du(i) \\
&= \underline{\alpha_d}\,\underline{u}(i-n) + \underline{\beta_d}\,\underline{y}(i-n) + Du(i) \\
&= \underline{\gamma_d\, ,_d}(i-1)
\end{aligned}
\tag{B33}
$$

where in the above equation $B' = B + M_d D$, $\overline{A} = A + M_d C$, and

$$
\underline{\gamma_d} = \begin{bmatrix} \underline{\alpha_d} & \underline{\beta_d} & D \end{bmatrix} \qquad \underline{,_d}(i-1) = \begin{bmatrix} \underline{u}(i-n) \\ \underline{y}(i-n) \\ u(i) \end{bmatrix}
\tag{B34}
$$

$$
\left.
\begin{aligned}
\underline{\alpha_d} &= \begin{bmatrix} \overline{Y}_0^{(1)} & \overline{Y}_1^{(1)} & \cdots & \overline{Y}_{n-1}^{(1)} \end{bmatrix} \\
\underline{\beta_d} &= \begin{bmatrix} \overline{Y}_0^{(2)} & \overline{Y}_1^{(2)} & \cdots & \overline{Y}_{n-1}^{(2)} \end{bmatrix}
\end{aligned}
\right\}
\tag{B35}
$$

The $nm \times 1$ input history vector $\underline{u}(i-n)$ and the $nq \times 1$ output history vector $\underline{y}(i-n)$ are defined as in equations (63), except $u(i)$ and $y(i)$ are now $m \times 1$ and $q \times 1$ vectors, respectively. Note that in the deadbeat scheme, the observer Markov parameters are solved directly from input-output data, and the actual system Markov parameters are then recovered simply as in equation (20).

Appendix C

The Mini-Mast Truss Structure

A model obtained by finite element analysis of the Mini-Mast truss structure (ref. 21) is used as an example to illustrate the identification algorithms developed in this paper. The mathematical model has the first two bending modes, with practically the same frequencies (0.8 Hz); the first torsional mode (4.3 Hz); and the second two bending modes (6.1 Hz), again with practically the same frequencies. The model considers two inputs and two outputs. The inputs are two torque wheels for the x and y axes, and the outputs are two displacement sensors mounted at the top of the structure as shown in figure C1. The system frequencies and the associated damping factors expressed as the real parts of the eigenvalues are listed in table C1.

Table C1. Damping and Frequencies of Truss Structure

Mode	Damping factor	Frequency, Hz
1	0.09	0.80
2	.09	.80
3	.33	4.36
4	.38	6.10
5	.39	6.16

The continuous-time system matrices are listed here. For ease of presentation, the matrices are subdivided and given below:

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix}
-8.918 \times 10^{-2} & -1.330 \times 10^{-4} & -5.035 & 4.756 \times 10^{-5} & 9.106 \times 10^{-4} \\
1.303 \times 10^{-4} & -8.912 \times 10^{-2} & -1.474 \times 10^{-4} & 5.032 & 1.309 \times 10^{-2} \\
5.035 & 1.540 \times 10^{-4} & -9.212 \times 10^{-2} & -1.293 \times 10^{-4} & -1.403 \times 10^{-3} \\
-4.100 \times 10^{-5} & -5.032 & 1.335 \times 10^{-4} & -9.205 \times 10^{-2} & -1.540 \times 10^{-2} \\
-3.238 \times 10^{-3} & -2.093 \times 10^{-3} & 3.540 \times 10^{-3} & 7.388 \times 10^{-3} & -3.251 \times 10^{-1} \\
4.008 \times 10^{-3} & -7.596 \times 10^{-3} & -4.048 \times 10^{-3} & 2.748 \times 10^{-3} & 27.420 \\
2.468 \times 10^{-2} & -9.535 \times 10^{-2} & -2.691 \times 10^{-2} & -1.040 \times 10^{-1} & 1.546 \times 10^{-3} \\
-9.585 \times 10^{-2} & -2.514 \times 10^{-2} & 1.043 \times 10^{-1} & -2.748 \times 10^{-2} & 3.791 \times 10^{-3} \\
2.660 \times 10^{-2} & -1.015 \times 10^{-1} & -2.617 \times 10^{-2} & -9.974 \times 10^{-2} & -3.283 \times 10^{-3} \\
-1.020 \times 10^{-1} & -2.627 \times 10^{-2} & 1.005 \times 10^{-1} & -2.567 \times 10^{-2} & 1.491 \times 10^{-2}
\end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1.549\times10^{-3} & 5.498\times10^{-3} & -1.999\times10^{-2} & -9.892\times10^{-3} & 3.740\times10^{-2} \\ 1.527\times10^{-2} & -2.110\times10^{-2} & -6.062\times10^{-3} & 3.859\times10^{-2} & 1.079\times10^{-2} \\ 1.214\times10^{-3} & 1.010\times10^{-2} & -3.766\times10^{-2} & -5.495\times10^{-3} & 2.055\times10^{-2} \\ -1.412\times10^{-2} & 3.884\times10-2 & 1.075\times10^{-2} & -2.179\times10^{-2} & -6.517\times10^{-3} \\ -27.420 & -9.447\times10^{-3} & 1.774\times10^{-2} & 1.114\times10^{-2} & -2.519\times10^{-2} \\ -3.330\times10^{-1} & -9.884\times10^{-3} & 2.238\times10^{-2} & 1.117\times10^{-2} & -2.125\times10^{-2} \\ 2.839\times10^{-3} & -3.763\times10^{-1} & 5.972\times10^{-1} & 38.364 & -1.010\times10^{-1} \\ -1.320\times10^{-2} & -5.956\times10^{-1} & -3.790\times10^{-1} & -4.656\times10^{-2} & 38.660 \\ -2.834\times10^{-3} & -38.364 & 4.638\times10^{-2} & -3.912\times10^{-1} & -5.969\times10^{-1} \\ -2.657\times10^{-3} & 1.011\times10^{-1} & -38.660 & 5.986\times10^{-1} & 3.943\times10^{-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 2.345\times10^{-3} & -1.996\times10^{-3} \\ -2.101\times10^{-3} & -2.360\times10^{-3} \\ -2.349\times10^{-3} & 1.999\times10^{-3} \\ -2.015\times10^{-3} & -2.364\times10^{-3} \\ -1.052\times10^{-4} & -2.488\times10^{-4} \\ 1.107\times10^{-4} & 2.455\times10^{-4} \\ 1.667\times10^{-3} & 9.519\times10^{-4} \\ -9.095\times10^{-4} & 1.554\times10^{-3} \\ 1.630\times10^{-3} & 9.180\times10^{-4} \\ -8.917\times10^{-4} & 1.509\times10^{-3} \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 0.000 & 0.000 \\ 0.000 & 0.000 \end{bmatrix}$$

where

$$C_1 = \begin{bmatrix} 1.119\times10^{-2} & 4.016\times10^{-3} & 1.122\times10^{-2} & -4.025\times10^{-3} & -9.167\times10^{-3} \\ -9.114\times10^{-3} & 7.620\times10^{-3} & -9.136\times10^{-3} & -7.639\times10^{-3} & -9.311\times10^{-3} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -9.177\times10^{-3} & -4.321\times10^{-4} & -2.448\times10^{-3} & 4.669\times10^{-4} & 2.393\times10^{-3} \\ -9.326\times10^{-3} & -2.427\times10^{-3} & 1.965\times10^{-3} & 2.423\times10^{-3} & -1.990\times10^{-3} \end{bmatrix}$$

In the numerical examples, the system model is discretized at a sampling frequency of 33.3 Hz.

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Figure C1. Mini-Mast structure showing the x and y torque wheel inputs TWA_x, TWA_y, and the displacement outputs D18A, D18B on bay 18 tip plane.