

Summary

A “robust closed-loop system” maintains desired stability and performance when the nominal system is subject to plant variations. Multivariable robustness analysis and robust control design tools take into account the uncertain nature of the plant models used in the design of feedback control systems. A fundamental research issue is to find the means of generating descriptions of model uncertainty consistent with robust multivariable analysis and design tools. One approach, based on sector stability theory, provides a frequency-domain description of the uncertainty associated with a particular model, given bounds on individual parameters in the model. This paper explores the application of this sector-based approach to the formulation of useful uncertainty descriptions for linear, time-invariant, multivariable systems. A review of basic sector properties and the sector-based approach are presented first. The sector-based approach is then applied to several general forms of parameter uncertainty to investigate its advantages and limitations. The results indicate that the sector uncertainty bound can be used effectively to evaluate the impact of parameter uncertainties on the frequency response of the design model. Inherent conservatism is a potential limitation of the sector-based approach, especially for highly dependent uncertain parameters. In addition, the representation of the system dynamics can affect the amount of conservatism reflected in the sector bound. Careful application of the method can help to reduce this conservatism, however, and the solution approach has some degrees of freedom that may be further exploited to reduce the conservatism.

1 Introduction

Uncertainties associated with plant models used in control design are known to limit attainable system performance. Attaining higher levels of performance for uncertain multi-input–multi-output systems is a key focus area in current controls research. One class of multivariable control analysis and design methods explicitly takes into account plant model uncertainties. These methods involve the generalization of Nyquist theory to multivariable systems by exploiting the properties of matrix norms. For example, methods of accounting for unstructured uncertainty use singular values as a basis for robustness tests that assess the likelihood of a system remaining stable in the face of particular uncertainty representations. These tests, however, require knowledge of the uncertainty magnitude over all frequencies to assess robustness accurately.

Consider the graphical interpretation of a typical robustness test for a linear system with unstructured multiplicative uncertainty reflected at the plant output depicted in figure 1 (Postlethwaite and Foo 1984). The upper curve is the reciprocal of the maximum singular value of the complementary sensitivity function. The lower curve is the maximum singular value of the uncertainty. The condition for robust stability is that the two curves do not intersect. The smallest distance between the two curves determines the level of robustness (i.e., the robustness margin). The frequency-dependent characterization of the uncertainty is therefore critical in accurately assessing system robustness. Other methods, such as those based on structured singular values, use the knowledge of the uncertainties in the form of weighting functions for the same reasons.

Attempts to use the complementary sensitivity function alone as a test for robustness are likely to give erroneous and misleading results. This is the case in figure 1. Point **a** corresponds to the point of minimum robustness when the uncertainty is assumed to be independent of frequency. Point **b**, on the other hand, corresponds to the point of minimum robustness when the frequency dependence is considered. A true test for robustness requires both curves. The upper curve alone only provides a relative robustness measure at best and an optimistic indication of the level of robustness and an erroneous indication of the critical frequency range at worst. (A nomenclature section appears after the references.)

Figure 1. Robustness test for unstructured uncertainty.

Design methods that explicitly take into account plant model uncertainties generally assume that descriptions of the uncertainties, for example, in the form of additive or multiplicative transfer function matrix blocks, are available as part of the plant model (Doyle and Stein 1981; Lehtomaki et al. 1984; Postlethwaite and Foo 1984; Ridgely and Banda 1986). Such representations of plant uncertainties are not usually part of the design model and methods of representing them are not well developed (Anderson and Schmidt 1989; Stein and Doyle 1978; Kwakernaak 1985). A fundamental research issue is finding the means to generate descriptions of plant model uncertainty consistent with robust multivariable analysis and design tools.

One approach to formulating uncertainty descriptions has been developed by Safonov (1978) and is based on sector stability theory (Zames 1966a, 1966b). The sector-based approach is very general and permits many forms of uncertainty (both linear and nonlinear) to be considered. It is also completely compatible with multivariable systems and singular-value-based robustness measures. The fundamental goal of the approach is to determine a bound on a given function of parameters for which values are within known bounds. The function may, for example, correspond to the frequency response of the plant, and the bounded parameters may correspond to various plant model quantities that, while uncertain, are known well enough to place a bound on their admissible values.

There are, however, many aspects of the sector-based approach that need to be better understood before it can be used with confidence. This paper employs sector stability theory concepts using Safonov's (1978) approach to develop an uncertainty description in a context that is consistent with linear, time-invariant, multi-input–multi-output (LTI-MIMO) systems. This study identifies key properties of the sector-based approach and highlights important implications of its application to robust control analysis methods. It is important to note that although this paper reviews the theoretical development performed by Safonov and applies it to linear multivariable systems, the emphasis is on the application (i.e., the specific issues associated with the application process and the resulting implications).

The paper begins with a review of the sector concept and the basic sector properties required to formulate the sector-based uncertainty representation in section 2. Section 3

begins by presenting a block diagram representation of uncertainty for LTI-MIMO systems that is consistent with the sector theory. The sector theory is then combined with the uncertainty representation to result in a sector-based uncertainty description that is compatible with multivariable robustness analysis and design methods. In section 4, this compatibility is demonstrated by combining the sector-based uncertainty description with one of the singular-value-based robustness conditions (Doyle and Stein 1981; Postlethwaite and Foo 1984). Finally, in section 5, the sector-based approach is applied to models that exhibit three classes of parameter uncertainty. The first class of uncertainty is associated with a MIMO plant model in which only parameters that appear in the numerators of the system transfer function matrix are uncertain. The second class of uncertainty is associated with a MIMO plant model in which only parameters appearing in the transfer function matrix denominator are uncertain. These examples illustrate a method for computing the multivariable sector bound and help identify and emphasize some key properties and limitations of sector-based uncertainty descriptions. The third class of uncertainty is for a single-input–single-output (SISO) plant model that is representative of real-world uncertainties that occur in the transfer function numerator and denominator simultaneously.

2 Sectors and Their Properties

The term *sector* comes from a geometric interpretation of some basic concepts of the stability theory from which sectors were developed (Zames 1966a, 1966b). These concepts are reviewed to understand the value of sector theory in describing plant model uncertainty and to identify its limitations.

Consider a real-valued function $f[u(t), \zeta]$ of the real variable $u(t)$ and parameter ζ ; the graph of this function is shown in figure 2. The variable u is itself a function of the independent variable t , which might typically be time. The graph of the function lies between the dotted lines with slopes labeled **a** and **b**. The region of the plane between the dotted lines containing the graph of the function forms a sector of the plane. If the function is known to lie inside the sector for all values of the parameter ζ , then the sector is a bound on all values of $f[u(t), \zeta]$. As a result, $f[u(t), \zeta]$, as a function of $u(t)$, can be characterized in terms of an expression involving the slope of the lines bounding the sector.

Figure 2. Geometric interpretation of sector bounds.

Define

$$\mathbf{c} \triangleq \frac{\mathbf{b} + \mathbf{a}}{2} \quad (1a)$$

$$\mathbf{r} \triangleq \frac{\mathbf{b} - \mathbf{a}}{2} \quad (\mathbf{b} \geq \mathbf{a}) \quad (1b)$$

to be the “center” and the “radius” of the sector, respectively. The fundamental property of the relationship between the function $f[u(t), \zeta]$ and the sector in which it is contained can be described mathematically in terms of \mathbf{c} and \mathbf{r} as follows:

$$|f[u(t), \zeta] - \mathbf{c}u(t)| \leq |\mathbf{r}u(t)| \quad (2)$$

In addition, equation (2) can be integrated over the range of f to give a *sector bound* involving the 2-norm, or

$$\|f[u(t), \zeta] - \mathbf{c}u(t)\|_2 \leq \|\mathbf{r}u(t)\|_2 \quad (3)$$

where the 2-norm for a real-valued scalar function is defined by

$$\|f(t)\|_2 \triangleq \int_{-\infty}^{\infty} f(t) f(t) dt \quad (4)$$

Any function that satisfies equation (3) is said to “lie in the interior of the sector with center \mathbf{c} and radius \mathbf{r} .”

The concept of the sector can be extended to the complex domain (s -plane) by applying Parseval’s formula (Papoulis 1962). Parseval’s formula equates the inner product of a real-valued function to the inner product of its Fourier transform. That is,

$$\int_{-\infty}^{\infty} f(t) f(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(-\omega) \hat{f}(\omega) d\omega \quad (5)$$

where \hat{f} is the Fourier transform of f ,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \exp(-j\omega t) f(t) dt \quad (6)$$

Therefore, in the complex domain the sector bound is simply

$$\|\hat{f}(s, \zeta) - \mathbf{c}\hat{u}(s)\|_2 \leq \|\mathbf{r}\hat{u}(s)\|_2 \quad (s = j\omega) \quad (7)$$

where the 2-norm for a complex-valued scalar function is defined by

$$\|\hat{f}(\omega)\|_2 \triangleq \int_{-\infty}^{\infty} \hat{f}^*(\omega) \hat{f}(\omega) d\omega \quad (8)$$

Now consider a mathematical generalization of the concept of a sector. Allow $\hat{f}(s)$ to be $\mathbf{F}(s)\hat{\mathbf{u}}(s)$, where $\hat{\mathbf{u}}(s)$ is a complex-valued vector and $\mathbf{F}(s)$ is a matrix of complex-valued functions. Then allow \mathbf{c} and \mathbf{r} to be expressed as matrices of complex-valued functions $\mathbf{C}(s)$ and $\mathbf{R}(s)$, respectively. The generalized form of the sector bound equation (eq. (7)) is then written as

$$\|\mathbf{F}(s) - \mathbf{C}(s)\|_2 \hat{\mathbf{u}}(s) \leq \|\mathbf{R}(s)\|_2 \hat{\mathbf{u}}(s) \quad (s = j\omega) \quad (9)$$

The physical interpretation of the sector is now obscured, but the same terminology is used when discussing the properties of equation (9), that is, center, radius, and sector bound. The sector bound in equation (9) is the form that leads to the application of sector theory to uncertainty modeling. However, some additional manipulations are required to relate the sector bound to recently established robustness concepts. The properties required to accomplish this are presented below.

2.1 Property 1

A significant advantage of the sector-based approach to bounding matrices of complex-valued functions is that the bound can be related to singular-value properties of the matrices. The 2-norm for functions of the type $\mathbf{A}(s)\mathbf{x}(s)$, where $\mathbf{A}(s)$ is a complex matrix and $\mathbf{x}(s)$ is a compatible vector, has the form

$$\|\mathbf{A}(s)\mathbf{x}(s)\|_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{x}^*(s) \mathbf{A}^*(s) \mathbf{A}(s) \mathbf{x}(s) ds \quad (10)$$

where the integrand is the Euclidean norm of $\mathbf{A}(s)\mathbf{x}(s)$, denoted by $\|\mathbf{A}(s)\mathbf{x}(s)\|_E$. By writing equation (9) in the form of equation (10) it can be readily shown that if

$$\|\mathbf{F}(s) \hat{\mathbf{u}}(s) - \mathbf{C}(s) \hat{\mathbf{u}}(s)\|_E \leq \|\mathbf{R}(s) \hat{\mathbf{u}}(s)\|_E \quad (s = j\omega) \quad (11)$$

for all $\omega \in [0, \infty)$, then equation (9) is satisfied as well. That is, if equation (11) is satisfied for every value of the independent variable s , then the integral of the left side is less than the integral of the right side. This is exactly the requirement for equation (9) to be satisfied.

Note that the minimum and maximum singular values of a matrix are related to the Euclidean norm by

$$\bar{\sigma}(\mathbf{A}) = \max_{\mathbf{x} \neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_E}{\|\mathbf{x}\|_E} \quad (12a)$$

$$\underline{\sigma}(\mathbf{A}) = \min_{\mathbf{x} \neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_E}{\|\mathbf{x}\|_E} \quad (12b)$$

where $\bar{\sigma}(\mathbf{A})$ and $\underline{\sigma}(\mathbf{A})$ indicate the maximum and minimum singular values of the matrix \mathbf{A} , respectively (Ridgely and Banda 1986). Applying this property to equation (11), one can see that equation (9) is guaranteed to be satisfied if

$$\bar{\sigma}[\mathbf{F}(s) - \mathbf{C}(s)] \leq \underline{\sigma}[\mathbf{R}(s)] \quad (s = j\omega) \quad (13)$$

for every $\omega \in [0, \infty)$ and $\|\mathbf{x}(s)\|_E \neq 0$. That is, if an upper bound on the left side of equation (11) is less than a lower bound on the right side, then equation (11) is satisfied. This form of the sector bound can be readily combined with established robustness properties, as shown subsequently.

2.2 Property 2

Consider the sector bound

$$\|f[u(t), \zeta] - \mathbf{c}u(t)\|_2 \leq \|\mathbf{r}u(t)\|_2 \quad (14)$$

If $f[u(t), \zeta]$ can be represented by $[\mathbf{c} + \delta c(t)]u(t)$, where $\delta c(t)$ represents a perturbation about \mathbf{c} , then equation (14) simplifies to

$$\|\delta c(t) u(t)\|_2 \leq \|\mathbf{r}u(t)\|_2 \quad (15)$$

One interpretation of this result is that if $\mathbf{c} + \delta c(t)$ lies in a sector with center \mathbf{c} and radius r , then $\delta c(t)$ lies in a sector with center at the origin (zero) and radius r . This indicates that the sector bound can be represented in a form that is independent of the nominal values of the parameters. In terms of the graphical interpretation of the sector (as in fig. 2), this corresponds to rotating the sector about the origin until it is symmetric with respect to the independent variable, as shown in figure 3. The sector bound for $\mathbf{c} + \delta c(t)$, equation (14) and figure 3(a), is therefore completely equivalent to the sector bound for $\delta c(t)$, equation (15) and figure 3(b).

(a) Sector for $\mathbf{c} + \delta c(t)$.

(b) Sector for $\delta c(t)$.

Figure 3. Equivalence of sector bounds.

2.3 Property 3

Another key property of sectors is that if there is a collection of sectors that are independent, they can be combined into a single matrix sector by concatenating them diagonally. Consider the collection of perturbations $c_i + \delta c_i$, which lie in sectors with centers \mathbf{c}_i and radii r_i such that none of the $c_i + \delta c_i$ depend on the other terms c_j and δc_j ($j \neq i$). Safonov and Athans (1978) have shown that, in this case,

$$\|(\mathbf{C} + \delta \mathbf{C}) \mathbf{x} - \mathbf{C} \mathbf{x}\|_2 \leq \|\mathbf{R} \mathbf{x}\|_2 \tag{16}$$

where

$$\mathbf{C} = \text{diag} [c_1, c_2, c_3, \dots] \quad (17a)$$

$$\delta\mathbf{C} = \text{diag} [\delta c_1, \delta c_2, \delta c_3, \dots] \quad (17b)$$

$$\mathbf{R} = \text{diag} [r_1, r_2, r_3, \dots] \quad (17c)$$

and \mathbf{x} is a compatible vector. This fact is important because it allows sector bounds on individual system parameters to be used to bound the entire system.

3 Sector Bounds for the System

The sector properties discussed above can be combined with an appropriately developed system representation to formulate a sector bound for the plant when the variations in its parameters have known sector bounds. The following development of the bound parallels the proof of Safonov's sector bound theorem (Safonov 1978; Safonov and Athans 1978, 1981). It is less rigorous than the proof, however, and is only intended to explain the key concepts.

Assume that the plant model is given such that

$$\mathbf{y}(s) = \mathbf{T}(s) \mathbf{u}(s) \quad (18)$$

where $\mathbf{y}(s)$ and $\mathbf{u}(s)$ are the system outputs and inputs, respectively, and $\mathbf{T}(s)$ is a linear, time-invariant, multivariable operator with uncertain parameters. The uncertainties cause perturbations in the plant response relative to the response of the nominal plant model. The response of the plant can therefore be written

$$\mathbf{y}(s) = \mathbf{L}_{yu}(s) \mathbf{u}(s) + \mathbf{d}(s) \quad (19)$$

where $\mathbf{L}_{yu}(s)$ is the transfer matrix for the nominal plant model and $\mathbf{d}(s)$ is the perturbation response. The perturbation response is directly dependent on the variations in the plant parameters due to the modeling uncertainty. The functional relationship between these quantities can be expressed as

$$\mathbf{d}(s) = f(s, \zeta) \mathbf{u}(s) \quad (20)$$

where $f(s, \zeta)$ represents perturbations in the system dynamics due to ζ , the parameter variations associated with uncertainties in the nominal plant model.

Assume that each uncertain parameter ζ_i is independent and lies in a sector centered at \mathbf{c}_i with radius r_i . Then according to property 3, the uncertainties can be combined into a perturbation matrix $\mathbf{C} + \delta\mathbf{C}$, with the sector bound given in equation (16). By property 2, an equivalent sector bound is

$$\|\delta\mathbf{C} \mathbf{x}\|_2 \leq \|\mathbf{R}\mathbf{x}\|_2 \quad (21)$$

In order to exploit this sector bound, the perturbation matrix $\delta\mathbf{C}$ must be isolated from the dynamics. That is, $\mathbf{d}(s)$ must be able to be expressed in the following form:

$$\mathbf{d}(s) = \delta\mathbf{L}_{yu}(s, \delta\mathbf{C}) \mathbf{u}(s) \quad (22)$$

where $\delta\mathbf{C}$ appears explicitly. The solution is to develop a feedback structure around $\delta\mathbf{C}$ that admits all forms of parameter variations associated with LTI-MIMO systems.

A structure that allows $\delta\mathbf{C}$ to be isolated from the plant dynamics is shown in figure 4. The perturbation matrix $\delta\mathbf{C}$ appears explicitly and the dynamic effects are characterized by

the transfer function matrices $\mathbf{L}_{eu}(s)$, $\mathbf{L}_{ev}(s)$, and $\mathbf{L}_{yv}(s)$. This structure is similar to that developed by Safonov (1978). The elements in the dashed box represent the portion of the system response associated with plant uncertainties and are referred to as the *uncertainty dynamics*. The transfer matrix $\mathbf{L}_{eu}(s)$ transforms the vector of system inputs $\mathbf{u}(s)$ into one compatible with $\delta\mathbf{C}$. The matrix $\mathbf{L}_{yv}(s)$ transforms the output of the perturbation block $\mathbf{v}(s)$ into the perturbation response $\mathbf{d}(s)$. The matrix $\mathbf{L}_{ev}(s)$ is required to allow the parameters in $\delta\mathbf{C}$ to appear in the denominator of the system transfer matrices. Without $\mathbf{L}_{ev}(s)$ it would only be possible to characterize the effects of uncertainties appearing in the numerator of the transfer matrix. This fact becomes evident when one attempts to characterize a denominator uncertainty without using $\mathbf{L}_{ev}(s)$, an exercise that is left for the reader.

Figure 4. General structure for uncertainty dynamics.

Consider the block diagram of the system representation in figure 4. It is evident from the diagram that

$$\mathbf{y}(s) = \mathbf{L}_{yu}(s)\mathbf{u}(s) + \mathbf{L}_{yv}(s)\mathbf{v}(s) \quad (23a)$$

$$\mathbf{e}(s) = \mathbf{L}_{eu}(s)\mathbf{u}(s) + \mathbf{L}_{ev}(s)\mathbf{v}(s) \quad (23b)$$

where $\mathbf{e}(s)$ and $\mathbf{v}(s)$ are the input and output of $\delta\mathbf{C}$, respectively. Since the perturbations are assumed to lie in a sector centered at the origin with radius \mathbf{R} (i.e., eq. (21) with \mathbf{x} replaced by $\mathbf{e}(s)$), the sector bound for $\delta\mathbf{C}$ can be written

$$\|\mathbf{v}(s)\|_E \leq \|\mathbf{R}(s)\mathbf{e}(s)\|_E \quad (24)$$

The objective is to obtain a sector bound on the overall system. This can be accomplished by replacing $\mathbf{v}(s)$ and $\mathbf{e}(s)$ in equation (24) with expressions in terms of $\mathbf{u}(s)$. The remainder of this section is devoted to the mathematical development required to isolate $\mathbf{u}(s)$. The main mathematical tool is the inner product defined by

$$\langle \mathbf{x}_1(s), \mathbf{x}_2(s) \rangle \triangleq \mathbf{x}_1^*(s) \mathbf{x}_2(s) \quad (25)$$

where $\mathbf{x}_1(s)$ and $\mathbf{x}_2(s)$ are compatible complex-valued vectors. The following discussion is limited to operations in the frequency domain and, for the sake of clarity, the argument s is deleted.

Substituting equation (23b) into equation (24) and expressing the Euclidean norm in terms of the inner product (i.e., $\|(\bullet)\|_E^2 = \langle (\bullet), (\bullet) \rangle$) results in

$$\langle \mathbf{v}, \mathbf{v} \rangle \leq \langle \mathbf{R}(\mathbf{L}_{eu}\mathbf{u} + \mathbf{L}_{ev}\mathbf{v}), \mathbf{R}(\mathbf{L}_{eu}\mathbf{u} + \mathbf{L}_{ev}\mathbf{v}) \rangle \quad (26)$$

which can be expanded with inner product operations to

$$0 \geq \langle \mathbf{v}, (\mathbf{I} - \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev}) \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{L}_{eu}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev} \mathbf{v} \rangle \\ - \langle \mathbf{v}, \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{eu} \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{L}_{eu}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{eu} \mathbf{u} \rangle \quad (27)$$

Define the following:

$$\mathbf{Q}_v \triangleq \mathbf{I} - \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev} \quad (28a)$$

$$\mathbf{R}_v \triangleq \mathbf{L}_{eu}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{eu} \quad (28b)$$

$$\mathbf{S}_v \triangleq \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{eu} \quad (28c)$$

Substituting these definitions into equation (27) results in the simpler expression

$$0 \geq \langle \mathbf{v}, \mathbf{Q}_v \mathbf{v} \rangle - \langle \mathbf{S}_v \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{S}_v \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{R}_v \mathbf{u} \rangle \quad (29)$$

Now define

$$\mathbf{T}_v \triangleq \mathbf{Q}_v^{-1} \mathbf{S}_v \quad (30)$$

and note that \mathbf{Q}_v usually has an inverse¹ and that $\mathbf{Q}_v^* = \mathbf{Q}_v$. Also note that the definition of \mathbf{T}_v has the following properties:

$$\langle (\mathbf{v} - \mathbf{T}_v \mathbf{u}), \mathbf{Q}_v (\mathbf{v} - \mathbf{T}_v \mathbf{u}) \rangle = \langle \mathbf{v}, \mathbf{Q}_v \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{T}_v^* \mathbf{Q}_v \mathbf{v} \rangle \\ - \langle \mathbf{v}, \mathbf{Q}_v \mathbf{T}_v \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{T}_v^* \mathbf{Q}_v \mathbf{T}_v \mathbf{u} \rangle \quad (31)$$

and

$$\mathbf{T}_v^* \mathbf{Q}_v = \mathbf{S}_v^* \mathbf{Q}_v^{-1} \mathbf{Q}_v = \mathbf{S}_v^* \quad (32)$$

Therefore, substituting \mathbf{T}_v into equation (29) and using the properties in equations (31) and (32) allows equation (29) to be simplified to

$$\langle (\mathbf{v} - \mathbf{T}_v \mathbf{u}), \mathbf{Q}_v (\mathbf{v} - \mathbf{T}_v \mathbf{u}) \rangle \leq \langle \mathbf{u}, \mathbf{P}_v \mathbf{u} \rangle \quad (33)$$

where

$$\mathbf{P}_v \triangleq \mathbf{R}_v + \mathbf{S}_v^* \mathbf{Q}_v^{-1} \mathbf{S}_v \quad (34)$$

In order to eliminate \mathbf{v} from equation (33) a special property of matrices is employed. Let \mathbf{A} and \mathbf{B} be appropriately dimensioned complex matrices, with \mathbf{A} being positive definite. If $\tilde{\mathbf{A}}$ is defined as

$$\tilde{\mathbf{A}} \triangleq (\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^*)^{-1} \quad (35)$$

then

$$\langle \mathbf{x}, \mathbf{B}^* \tilde{\mathbf{A}} \mathbf{B} \mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle \quad (36)$$

This is lemma A1 (Safonov 1978) and it is proved in appendix A.

Let $\mathbf{A} = \mathbf{Q}_v$ and $\mathbf{B} = \mathbf{L}_{yv}$; then,

$$\langle \mathbf{x}, \mathbf{L}_{yv}^* (\mathbf{L}_{yv} \mathbf{Q}_v^{-1} \mathbf{L}_{yv}^*)^{-1} \mathbf{L}_{yv} \mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{Q}_v \mathbf{x} \rangle \quad (37)$$

¹ A condition for the existence of \mathbf{Q}_v^{-1} is discussed in section 5.1.

If \mathbf{x} in equation (37) is replaced by $(\mathbf{v} - \mathbf{T}_v \mathbf{u})$ and compared with equation (33), then one finds that

$$\langle \mathbf{L}_{yv} (\mathbf{v} - \mathbf{T}_v \mathbf{u}), (\mathbf{L}_{yv} \mathbf{Q}_v^{-1} \mathbf{L}_{yv}^*)^{-1} \mathbf{L}_{yv} (\mathbf{v} - \mathbf{T}_v \mathbf{u}) \rangle \leq \langle \mathbf{u}, \mathbf{P}_v \mathbf{u} \rangle \quad (38)$$

Substituting equation (23a) into equation (38) results in

$$\langle [\mathbf{y} - (\mathbf{L}_{yu} + \mathbf{L}_{yv} \mathbf{T}_v) \mathbf{u}], (\mathbf{L}_{yv} \mathbf{Q}_v^{-1} \mathbf{L}_{yv}^*)^{-1} [\mathbf{y} - (\mathbf{L}_{yu} + \mathbf{L}_{yv} \mathbf{T}_v) \mathbf{u}] \rangle \leq \langle \mathbf{u}, \mathbf{P}_v \mathbf{u} \rangle \quad (39)$$

Note that equation (39) is satisfied whenever equation (24) is satisfied. All dependence on \mathbf{e} and \mathbf{v} has been removed. The information about the sector bounds on the system parameters is now completely contained in \mathbf{P}_v , \mathbf{Q}_v , and \mathbf{T}_v . All that remains is to simplify equation (39) into a form consistent with the sector bound equation.

Define

$$\mathbf{T}_n \triangleq \mathbf{L}_{yu} + \mathbf{L}_{yv} \mathbf{T}_v \quad (40a)$$

$$\mathbf{Q}_n \triangleq (\mathbf{L}_{yv} \mathbf{Q}_v^{-1} \mathbf{L}_{yv}^*)^{-1} \quad (40b)$$

$$\mathbf{P}_n \triangleq \mathbf{P}_v \quad (40c)$$

Substitution of equations (40) into equation (39) results in

$$\langle (\mathbf{y} - \mathbf{T}_n \mathbf{u}), \mathbf{Q}_n (\mathbf{y} - \mathbf{T}_n \mathbf{u}) \rangle \leq \langle \mathbf{u}, \mathbf{P}_n \mathbf{u} \rangle \quad (41)$$

Now, since \mathbf{Q}_n and \mathbf{P}_n are Hermitian matrices, they can be factored² so that equation (41) can be written as

$$\|\mathbf{Q}_n^{1/2} (\mathbf{y} - \mathbf{T}_n \mathbf{u})\|_2 \leq \|\mathbf{P}_n^{1/2} \mathbf{u}\|_2 \quad (42)$$

Substituting equation (18) into equation (42) results in

$$\|\mathbf{Q}_n^{1/2} (\mathbf{T} - \mathbf{T}_n) \mathbf{u}\|_2 \leq \|\mathbf{P}_n^{1/2} \mathbf{u}\|_2 \quad (43)$$

where, in summary,

$$\mathbf{T}_n \triangleq \mathbf{L}_{yu} + \mathbf{L}_{yv} (\mathbf{I} - \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev})^{-1} \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{eu} \quad (44a)$$

$$\mathbf{P}_n \triangleq \mathbf{L}_{eu}^* \mathbf{R}^* (\mathbf{I} - \mathbf{R} \mathbf{L}_{ev} \mathbf{L}_{ev}^* \mathbf{R}^*)^{-1} \mathbf{R} \mathbf{L}_{eu} \quad (44b)$$

$$\mathbf{Q}_n \triangleq [\mathbf{L}_{yv} (\mathbf{I} - \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev})^{-1} \mathbf{L}_{yv}^*]^{-1} \quad (44c)$$

and where the expression for \mathbf{P}_n was obtained by use of the matrix inversion lemma (pp. 48–49 of Brogan 1974).

Equation (43) is in the form of a sector bound that is slightly more general than that discussed previously. (See eq. (9).) Notice that the left side has the added weighting $\mathbf{Q}_n^{1/2}$, which did not appear in the earlier form. This causes little difficulty, as shown subsequently. Therefore, equation (43) is a generalized sector bound on the overall plant response given the sector bounds on the uncertain plant parameters.

² The factoring method used in the examples is the Cholesky decomposition (Strang 1980).

4 Robustness Analysis Using Sector Bounds

The sector bound shown in equation (43) can be used to compute an upper bound on \mathbf{T} (i.e., an uncertainty bound) that is consistent with LTI-MIMO robustness analysis and robust control design methods. Begin by considering a singular-value form of the sector bound from equation (43):

$$\bar{\sigma} \left[\mathbf{Q}_n^{1/2} (\mathbf{T} - \mathbf{T}_n) \right] \leq \underline{\sigma} \left[\mathbf{P}_n^{1/2} \right] \quad (s = j\omega; \omega \in [0, \infty)) \quad (45)$$

If equation (45) is satisfied, then equation (43) is satisfied as well. Equation (45) can be expanded through the use of singular-value properties (Ridgely and Banda 1986) as follows:

$$\begin{aligned} \bar{\sigma} \left[\mathbf{Q}_n^{1/2} (\mathbf{T} - \mathbf{T}_n) \right] &\leq \underline{\sigma} \left[\mathbf{P}_n^{1/2} \right] \\ \Leftrightarrow \bar{\sigma} [\mathbf{T} - \mathbf{T}_n] &\leq \underline{\sigma} \left[\mathbf{P}_n^{1/2} \right] \underline{\sigma} \left[\mathbf{Q}_n^{-1/2} \right] \end{aligned} \quad (46)$$

since, for compatible arbitrary matrices \mathbf{A} and \mathbf{B} ,

$$\bar{\sigma} [\mathbf{AB}] \leq \bar{\sigma} [\mathbf{A}] \bar{\sigma} [\mathbf{B}]$$

and for definite \mathbf{A} ,

$$\underline{\sigma} \left[\mathbf{A}^{-1} \right] = \frac{1}{\bar{\sigma} [\mathbf{A}]}$$

Thus, if equation (46) is satisfied for all $s = j\omega$ ($\omega \in [0, \infty)$), then the sector bound (eq. (43)) is also satisfied. Equation (46) is a general singular-value representation of a sector-based plant model uncertainty bound. It is in a form that can be readily combined with recently established singular-value-based robustness tests.

The combination of the uncertainty bound with a robustness test requires the choice of a specific characterization of the plant model uncertainty. For the purposes of demonstration, consider a typical formulation with additive uncertainty as shown in figure 5. This uncertainty formulation is used throughout the remainder of this paper. Other uncertainty formulations (e.g., multiplicative uncertainty at the plant input or output) give similar and analogous results.

Figure 5. Additive plant uncertainty.

A condition for guaranteed stability of the closed-loop system depicted in figure 5 subject to the uncertainty Δ is

$$\bar{\sigma}[\Delta] < \frac{1}{\bar{\sigma}[\mathbf{I} + \mathbf{K}\mathbf{G}]^{-1} \mathbf{K}} \quad (s = j\omega; \omega \in [0, \infty)) \quad (47)$$

where \mathbf{G} is the nominal plant transfer matrix and \mathbf{K} is the feedback compensation (Doyle and Stein 1981; Barrett 1980; Postlethwaite and Foo 1984). The plant uncertainty represented by Δ may not actually be known, but a quantitative upper bound on $\bar{\sigma}[\Delta]$ can be obtained with the sector-based approach if the uncertain parameters in the plant model have known bounds. The objective is therefore to obtain an upper bound on $\bar{\sigma}[\Delta]$ from the sector bound equation.

Simple block diagram manipulations indicate that the open-loop system representation can be written as

$$\mathbf{T} = \mathbf{G} + \Delta \quad (48)$$

A similar expression for \mathbf{T}_n in equation (46) can be written:

$$\mathbf{T}_n = \mathbf{G} + \tilde{\Delta} \quad (49)$$

where \mathbf{G} corresponds to the nominal plant transfer matrix, \mathbf{L}_{yu} in equation (44a), and $\tilde{\Delta}$ corresponds to the remaining term in equation (44a). Note that $\tilde{\Delta}$ is a purely mathematical quantity that in this case is analogous to Δ . Substituting these expressions into equation (46) results in the equivalent sector bound equation

$$\bar{\sigma}[\Delta - \tilde{\Delta}] \leq \underline{\sigma}[\mathbf{P}_n^{1/2}] \underline{\sigma}[\mathbf{Q}_n^{-1/2}] \quad (50)$$

Now consider the single-input–single-output form of equation (50) where Δ and $\tilde{\Delta}$ are complex scalars and \mathbf{P}_n and \mathbf{Q}_n are replaced by the scalars p and q . Equation (50) takes on the simplified form

$$|\Delta - \tilde{\Delta}| \leq |(p/q)^{1/2}| \quad (51)$$

A geometric interpretation of this equation indicates that Δ lies inside a circle centered at $\tilde{\Delta}$ with radius $|(p/q)^{1/2}|$, as depicted in figure 6.

Figure 6. Geometric interpretation of sector bound.

Let a real parameter δ be chosen such that

$$\delta = |\tilde{\Delta}| + |(p/q)^{1/2}| \quad (52)$$

Clearly δ is an upper bound on $|\Delta|$ (i.e., $\delta \geq |\Delta|$). Generalizing this to matrices implies that if

$$\delta = \bar{\sigma} [\tilde{\Delta}] + \bar{\sigma} [\mathbf{P}_n^{1/2}] \bar{\sigma} [\mathbf{Q}_n^{-1/2}] \quad (53)$$

then δ is an upper bound on $\bar{\sigma}[\Delta]$, which can be proved as follows. Let α be a matrix such that $\underline{\sigma}[\alpha] \equiv \delta$. Then,

$$\underline{\sigma}[\alpha] - \bar{\sigma} [\tilde{\Delta}] = \bar{\sigma} [\mathbf{P}_n^{1/2}] \bar{\sigma} [\mathbf{Q}_n^{-1/2}] \quad (54)$$

$$\Rightarrow \underline{\sigma} [\alpha - \tilde{\Delta}] \geq \bar{\sigma} [\mathbf{P}_n^{1/2}] \bar{\sigma} [\mathbf{Q}_n^{-1/2}] \quad (55)$$

since for compatible arbitrary matrices \mathbf{A} and \mathbf{B} (Ridgely and Banda 1986),

$$\underline{\sigma}[\mathbf{A} - \mathbf{B}] \geq \underline{\sigma}[\mathbf{A}] - \bar{\sigma}[\mathbf{B}]$$

Combining equation (55) with equation (50) gives

$$\underline{\sigma} [\alpha - \tilde{\Delta}] \geq \bar{\sigma} [\mathbf{P}_n^{1/2}] \bar{\sigma} [\mathbf{Q}_n^{-1/2}] \geq \underline{\sigma} [\mathbf{P}_n^{1/2}] \underline{\sigma} [\mathbf{Q}_n^{-1/2}] \geq \bar{\sigma} [\Delta - \tilde{\Delta}] \quad (56)$$

As a result, α is a bound on the admissible uncertainty. This implies that

$$\bar{\sigma}[\Delta] \leq \underline{\sigma}[\alpha] = \delta \quad (57)$$

so

$$\bar{\sigma}[\Delta] \leq \bar{\sigma} [\tilde{\Delta}] + \bar{\sigma} [\mathbf{P}_n^{1/2}] \bar{\sigma} [\mathbf{Q}_n^{-1/2}] \quad (58)$$

The right side of equation (58) is a sector-based uncertainty bound that is compatible with the singular-value-based robustness condition. The same expression results for other uncertainty representations (e.g., multiplicative uncertainty reflected at the plant input). The only difference is that $\tilde{\Delta}$ has a different interpretation that depends on the uncertainty representation.

Equation (58) can be used in the robustness condition, equation (47), in place of $\bar{\sigma}[\Delta]$ as shown in equation (59) to provide a robustness test that can be readily computed:

$$\bar{\sigma} [\tilde{\Delta}] + \bar{\sigma} [\mathbf{P}_n^{1/2}] \bar{\sigma} [\mathbf{Q}_n^{-1/2}] < \frac{1}{\bar{\sigma} [\mathbf{I} + \mathbf{GK}]^{-1} \mathbf{K}} \quad (59)$$

Equation (59) can be used as a sufficient robustness condition to verify that known sector bounded uncertainties will not destabilize the closed-loop system. The implications of equation (59) are the same as those of equation (47). That is, if equation (59) is satisfied, then the closed-loop system is guaranteed to be stable for the modeled plant variations. Violation of equations (47) and (59) does not, however, imply that the closed-loop system is necessarily unstable (Doyle and Stein 1981).

An interpretation of this result is illustrated in figure 7. The upper curve is a plot of the reciprocal of the maximum singular value of $[\mathbf{I} + \mathbf{GK}]^{-1} \mathbf{K}$. The lower curve is a plot of the maximum singular value of the “exact” uncertainty Δ . The middle curve is a plot of the right side of equation (58), and since it is an upper bound on the actual uncertainty, it is

used in equation (59) to test for system stability subject to the sector bounded uncertainties. Equation (59) is satisfied as long as the upper and middle curves do not intersect. This condition is potentially more conservative than equation (47), however, since equation (59) may be violated while equation (47) is still satisfied. In graphical terms, the upper and middle curves may intersect without causing the lower and upper curves to intersect.

Figure 7. Interpretation of sector-based robustness condition.

5 Limitations and Applications

In this section limitations in using the uncertainty bound in equation (58) are discussed. Some of the limitations are directly observable from the development itself while others become more apparent when the approach is applied to typical forms of parameter uncertainties. Example problems are, therefore, used to illustrate these limitations and at the same time to demonstrate the solution technique.

5.1 Limitations

A couple of conditions on the applicability of the uncertainty bound are linked to equations (44). In order for the system sector bound to exist the inverse of

$$\mathbf{Q}_v \triangleq \mathbf{I} - \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev}$$

(eq. (28a)) must exist since \mathbf{T}_n , \mathbf{P}_n , and \mathbf{Q}_n from equations (44) all depend on the existence of \mathbf{Q}_v^{-1} , either directly or indirectly. In addition, \mathbf{Q}_n only exists if $\mathbf{L}_{yv} \mathbf{Q}_v^{-1} \mathbf{L}_{yv}^*$ has an inverse. Simple tests for the existence of these inverses are the following:

1. \mathbf{Q}_v will have an inverse if $\bar{\sigma}[\mathbf{R} \mathbf{L}_{ev}] < 1$. (Note that this is only a sufficient condition.)
2. $\mathbf{L}_{yv} \mathbf{Q}_v^{-1} \mathbf{L}_{yv}^*$ will have an inverse if \mathbf{Q}_v has an inverse and if \mathbf{L}_{yv} is an $m \times n$ matrix, where $m \leq n$ (m is the number of outputs and n is the number of uncertain parameters) and \mathbf{L}_{yv} is full rank (i.e., rank m).

Proofs of these conditions are presented in appendix A. The consequences of these conditions are fairly minor. A violation of the first condition implies that sector radii are sufficiently large that

the maximum singular value of the matrix product \mathbf{RL}_{ev} exceeds unity. This can be remedied by using a different set of physical units. For example, assume a parameter is represented in units of degrees per second and the sector radius is 100. Converting the units to radians per second reduces the sector radius to 0.556, which is likely to cause the existence condition to be satisfied.

The second condition is violated when there are less uncertainties than outputs or when an output of the system is not affected by any of the modeled uncertainties. It is not likely for realistic systems to contain less uncertainties than outputs, since there are typically a large number of uncertain parameters. However, if there are fewer uncertain parameters than outputs, the parameter must be repeated until the condition is satisfied. In the latter situation, if a transfer function in the system does not have any uncertainty, it should not be included in the computation of the sector bound. Therefore, the remedy is to reformulate the problem such that only outputs that are dependent on uncertain parameters are considered.

A slightly stronger limitation of the sector theory is that it only applies when the true plant is stable for all perturbations inside the parameter sectors. This is because the relationship between the frequency-domain and the time-domain representation of the system (via Parseval's formula) is only valid for stable systems. This is not too disconcerting for robustness applications because the robustness conditions (e.g., eq. (47)) effectively limit the perturbations to be nondestabilizing (open loop) since the conditions require that the number of right half-plane poles of the open-loop system not change under admissible perturbations (Doyle and Stein 1981).

Yet another limitation is associated with the conservatism introduced by sector bounds. In the frequency domain, if a single parameter α is known to lie in a sector with center β and radius ρ , then α can take on any complex value inside the circle, as shown in figure 8. Thus, even if α is known to be a strictly real-valued parameter, bounding it within a frequency-domain sector admits complex-valued perturbations $\delta\alpha$. As a result, conservatism is introduced into the system sector bound when uncertain parameters are strictly real-valued parameters (or otherwise conservatively bounded within a circle). Additional conservatism is introduced by the singular-value form of the sector bound since it is obtained through use of inequality properties of singular values.

Figure 8. Sector bound for real parameter.

Conservatism is also introduced because of the restrictions on cross coupling of the uncertain parameters. Recall that by property 3 the uncertain parameter matrix $\delta\mathbf{C}$ is required to be diagonal. This allows the individual parameter sectors to be combined into a single-matrix sector. (See eqs. (16) and (17).) However, this also requires that each uncertain parameter

be treated as if it is independent of the other uncertain parameters. Also, the fact that $\delta\mathbf{C}$ is diagonal limits the ways by which the system can be represented to formulate the system sector bound. This conservatism can best be illustrated by examples.

The following examples serve to identify some of the factors that influence the conservatism of the sector-based uncertainty bound. In addition, the examples demonstrate a method of solving for the uncertainty dynamics matrices for general plant parameter uncertainties.

5.2 Numerator Uncertainty for a Multivariable System

In this example a simple two-input-two-output system is considered to have the transfer matrix

$$\mathbf{G}(s) = \frac{1}{s^2 + \mathbf{d}s + \mathbf{e}} \begin{bmatrix} s + \mathbf{a} & \mathbf{a} \\ s + \mathbf{b} & s^2 + \mathbf{a}s + \mathbf{c} \end{bmatrix} \quad (60)$$

where each of the numerator coefficients is uncertain but is known to lie in a sector. That is,

$$\mathbf{a} = \underline{\mathbf{a}} + \mathbf{a}' \quad \mathbf{b} = \underline{\mathbf{b}} + \mathbf{b}' \quad \mathbf{c} = \underline{\mathbf{c}} + \mathbf{c}' \quad (61)$$

where, for example, $\underline{\mathbf{a}}$ is the nominal value of \mathbf{a} and \mathbf{a}' is the perturbation of \mathbf{a} . The sector bounds on the coefficients are such that the perturbation value is centered at the origin with radius r (e.g., \mathbf{a}' is in a sector centered at zero with radius $r_{\mathbf{a}}$).

Denote the system inputs by $\mathbf{u}(s)$ and the system outputs by $\mathbf{y}(s)$. The system frequency-domain input-output equation can then be written as

$$(s^2 + \mathbf{d}s + \mathbf{e}) \mathbf{y}(s) = \begin{bmatrix} s + \underline{\mathbf{a}} & \underline{\mathbf{a}} \\ s + \underline{\mathbf{b}} & s^2 + \underline{\mathbf{a}}s + \underline{\mathbf{c}} \end{bmatrix} \mathbf{u}(s) + \begin{bmatrix} \mathbf{a}' & \mathbf{a}' \\ \mathbf{b}' & \mathbf{a}'s + \mathbf{c}' \end{bmatrix} \mathbf{u}(s) \quad (62)$$

The perturbation response (as in eq. (19)) due to uncertainties is simply

$$\mathbf{d}(s) = \frac{1}{s^2 + \mathbf{d}s + \mathbf{e}} \begin{bmatrix} \mathbf{a}' & \mathbf{a}' \\ \mathbf{b}' & \mathbf{a}'s + \mathbf{c}' \end{bmatrix} \mathbf{u}(s) \quad (63)$$

Block diagram manipulations of the uncertainty dynamics, $\delta\mathbf{L}_{yu}$ in figure 4, indicate that the disturbance can also be expressed as

$$\mathbf{d}(s) = \mathbf{L}_{yv}(s) \delta\mathbf{C} \mathbf{L}_{eu}(s) \mathbf{u}(s) \quad (64)$$

Notice that $\mathbf{L}_{ev}(s)$ is zero because there are no denominator uncertainties.

Combining equations (64) and (65) gives the equation that governs how $\delta\mathbf{C}$, $\mathbf{L}_{eu}(s)$, and $\mathbf{L}_{yv}(s)$ may be chosen:

$$\mathbf{L}_{yv}(s) \delta\mathbf{C} \mathbf{L}_{eu}(s) = \frac{1}{s^2 + \mathbf{d}s + \mathbf{e}} \begin{bmatrix} \mathbf{a}' & \mathbf{a}' \\ \mathbf{b}' & \mathbf{a}'s + \mathbf{c}' \end{bmatrix} \quad (65)$$

The solution of equation (65) is not unique. There is considerable freedom in choosing the transfer matrices in the uncertainty dynamics since the scalar equations that govern the choice of the matrix elements are underspecified. The governing scalar equations are also nonlinear, a fact that complicates their solution.

The first step in solving equation (65) is the choice of the $\delta\mathbf{C}$ matrix. The simplest way of choosing $\delta\mathbf{C}$ is to assume that each appearance of an uncertain parameter is associated with a different sector. For example, since the parameter \mathbf{a} appears three times in the input-output

equation, each of the three is assumed to be a different parameter and so is associated with an independent sector. The perturbation matrix and the associated sector radius matrix are therefore

$$\delta \mathbf{C} = \text{diag} [\mathbf{a}', \mathbf{a}', \mathbf{a}', \mathbf{b}', \mathbf{c}'] \quad (66a)$$

$$\mathbf{R} = \text{diag} [r_{\mathbf{a}}, r_{\mathbf{a}}, r_{\mathbf{a}}, r_{\mathbf{b}}, r_{\mathbf{c}}] \quad (66b)$$

One would expect the system sector bound resulting from this choice for $\delta \mathbf{C}$ to be overly conservative because, in effect, \mathbf{a} is allowed to take on a different value every time it appears in the system input-output equation. This issue is addressed subsequently. For now, consider the process by which $\mathbf{L}_{eu}(s)$ and $\mathbf{L}_{yv}(s)$ can be chosen subject to the choice of $\delta \mathbf{C}$. Let

$$\mathbf{L}_{yv}(s) = \frac{\mathbf{N}_{yv}(s)}{d_{yv}(s)} \quad \mathbf{L}_{eu}(s) = \frac{\mathbf{N}_{eu}(s)}{d_{eu}(s)} \quad (67)$$

where $\mathbf{N}_{yv}(s)$ and $\mathbf{N}_{eu}(s)$ are matrices of numerator polynomials and $d_{yv}(s)$ and $d_{eu}(s)$ are denominator polynomials. Recall from equation (23a) that

$$\mathbf{y}(s) = \mathbf{L}_{yu}(s) \mathbf{u}(s) + \mathbf{L}_{yv}(s) \mathbf{v}(s)$$

Thus, if $d_{yv}(s)$ is chosen to be the nominal system denominator polynomial, then both $\mathbf{L}_{yu}(s)$ and $\mathbf{L}_{yv}(s)$ will have a common denominator. Therefore, let

$$d_{yv}(s) = s^2 + \mathbf{d}s + \mathbf{e} \quad (68)$$

Equation (63) and the choice for $d_{yv}(s)$ then require that

$$d_{eu}(s) \equiv 1 \quad (69)$$

For dimensional consistency $\mathbf{N}_{yv}(s)$ must be a 2×5 matrix of polynomials and $\mathbf{N}_{eu}(s)$ must be a 5×2 matrix of polynomials. Therefore, let

$$\mathbf{N}_{yv}(s) = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} \end{bmatrix} \quad (70a)$$

$$\mathbf{N}_{eu}^T(s) = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} & m_{51} \\ m_{12} & m_{22} & m_{32} & m_{42} & m_{52} \end{bmatrix} \quad (70b)$$

Substituting equations (70) and (66a) into equation (65), expanding the elements, and grouping like terms results in a system of nonlinear equations for each uncertain parameter. This is true regardless of how many times a parameter appears in the system equations. Therefore, there are three systems of equations to solve in this example, one associated with each of the uncertain parameters \mathbf{a} , \mathbf{b} , and \mathbf{c} .

System for \mathbf{a} :

$$\left. \begin{aligned} 1 &= l_{11}m_{11} + l_{12}m_{21} + l_{13}m_{31} \\ 1 &= l_{11}m_{12} + l_{12}m_{22} + l_{13}m_{32} \\ 0 &= l_{21}m_{11} + l_{22}m_{21} + l_{23}m_{31} \\ s &= l_{21}m_{12} + l_{22}m_{22} + l_{23}m_{32} \end{aligned} \right\} \quad (71a)$$

System for \mathbf{b} :

$$\left. \begin{array}{l} 0 = l_{14}m_{41} \quad 0 = l_{14}m_{42} \\ 1 = l_{24}m_{41} \quad 0 = l_{24}m_{42} \end{array} \right\} \quad (71b)$$

System for \mathbf{c} :

$$\left. \begin{array}{l} 0 = l_{15}m_{51} \quad 0 = l_{15}m_{52} \\ 0 = l_{25}m_{51} \quad 1 = l_{25}m_{52} \end{array} \right\} \quad (71c)$$

The solutions to these systems of equations are not unique. However, since the elements of $\mathbf{N}_{yv}(s)$ and $\mathbf{N}_{eu}(s)$ are numerator polynomials, practical restrictions can be put on them that make the solutions tractable. Require the elements m_{ij} and n_{ij} to be *minimal polynomials* (i.e., they only involve either nonnegative integer powers of s or zero) and to be of the lowest possible order and least possible number of terms. For example, m_{ij} or n_{ij} will not be allowed to involve terms like s^{-1} or $s^{0.5}$.

With these restrictions, reasonable solutions of the systems of equations for nonrepeated parameters, such as \mathbf{b} and \mathbf{c} , are quite simple. A solution to the system for \mathbf{b} is

$$l_{14} = 0 \quad l_{24} = 1 \quad m_{41} = 1 \quad m_{42} = 0 \quad (72)$$

and a solution to the system for \mathbf{c} is

$$l_{15} = 0 \quad l_{25} = 1 \quad m_{51} = 0 \quad m_{52} = 1 \quad (73)$$

The solution to the system of equations for repeated parameters, such as \mathbf{a} , is slightly more involved since, even with the restrictions, there are many more possible solutions. The system for \mathbf{a} can be solved by making some ‘‘inspired guesses.’’ Let

$$l_{11} = l_{22} = 1 \quad l_{12} = l_{21} = 0 \quad l_{13} = l_{23} = 0 \quad m_{22} = s \quad (74)$$

The remaining terms in the system for \mathbf{a} then become

$$m_{11} = m_{12} = 1 \quad m_{21} = 0 \quad m_{31} = m_{32} = \text{Arbitrary} \quad (75)$$

Since m_{31} and m_{32} are arbitrary let them both be 1, for example. With these values for the numerator terms $\mathbf{N}_{yv}(s)$ and $\mathbf{N}_{eu}(s)$, the complete uncertainty dynamics representation becomes

$$\delta\mathbf{C} = \text{diag}[\mathbf{a}', \mathbf{a}', \mathbf{a}', \mathbf{b}', \mathbf{c}'] \quad \mathbf{L}_{eu}^T(s) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & s & 1 & 0 & 1 \end{bmatrix} \quad (76a)$$

$$\mathbf{L}_{yv}(s) = \frac{1}{s^2 + \mathbf{d}s + \mathbf{e}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{L}_{ev}(s) = [0]_{5 \times 5} \quad (76b)$$

with the associated sector radius matrix \mathbf{R} from equation (66b).

Note that the third column of $\mathbf{L}_{yv}(s)$ is zero, and so the third appearance of the perturbation \mathbf{a}' does not influence the plant responses. This implies that the third appearance of the perturbation \mathbf{a}' is superfluous and may add unnecessary conservatism to the resulting system sector bound. It can be dropped from $\delta\mathbf{C}$ without affecting the ability to appropriately represent the uncertainty dynamics $\delta\mathbf{L}_{yu}(s)$.

A key question associated with the solution to equations (71a) presented above needs to be addressed: Is there actually additional conservatism in the sector bound associated with an excessively repeated perturbation parameter? This question can be answered by considering a

numerical example based on the solution presented in equations (76). Consider an alternate choice for $\delta\mathbf{C}$ in which the number of repetitions of \mathbf{a}' has been reduced by one. A solution for the uncertainty dynamics is

$$\delta\mathbf{C} = \text{diag} [\mathbf{a}', \mathbf{a}', \mathbf{b}', \mathbf{c}'] \quad \mathbf{L}_{eu}^T(s) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & s & 0 & 1 \end{bmatrix} \quad (77a)$$

$$\mathbf{L}_{yv}(s) = \frac{1}{s^2 + \mathbf{d}s + \mathbf{e}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{L}_{ev}(s) = [0]_{4 \times 4} \quad (77b)$$

with an appropriately reduced sector radius matrix \mathbf{R} .

One would expect the uncertainty dynamics in equations (76) to result in a more conservative uncertainty bound than that from equations (77) since, in equations (76), \mathbf{a}' can effectively take on three different values simultaneously rather than only two, as in equations (77). The conservatism associated with the added degree of freedom is demonstrated in figure 9. These bounds are generated with equation (58)³, with the values for the system parameters given by

$$\begin{aligned} \underline{\mathbf{a}} &= 5.4 & \underline{\mathbf{b}} &= 10.0 & \underline{\mathbf{c}} &= 25.25 & \mathbf{d} &= 10.0 & \mathbf{e} &= 100.0 \\ \mathbf{r}_a &= 4.5 & \mathbf{r}_b &= 2.0 & \mathbf{r}_c &= 5.0 \end{aligned}$$

Even though the bound obtained from equations (77) is less conservative than that from equations (76), it still exhibits conservatism. Some of the conservatism is because \mathbf{a}' can still simultaneously take on two different values inside its sector, when in reality it can only have one value. Additional conservatism is introduced when \mathbf{a}' is actually real valued, since the sector bound assumes complex perturbations. This conservatism is evident when the sector bounds are compared with an exact bound for the assumption of only real-valued perturbations, as in figure 9.

Figure 9. Conservatism of uncertainty bound due to perturbation representation.

³ A listing of the sector bound algorithm is presented in appendix B.

The exact bound is obtained by using a constrained optimization procedure (Jacob 1972). The procedure uses a nongradient search method to maximize the singular value of the additive uncertainty matrix, Δ in figure 5, associated with the uncertain parameters. In this case Δ corresponds to the transfer function matrix between $\mathbf{d}(s)$ and $\mathbf{u}(s)$ (eq. (63)). The uncertainties are represented as real perturbations about the nominal values of the uncertain parameters. The maximum variations are symmetric about the nominal values and equal to the sector radii.

As shown above, conservatism is introduced into the sector bound when a perturbation parameter is repeated more times than required. Therefore, it would be desirable to be able to identify when a parameter is excessively repeated so that the conservatism can be reduced. Alternatively, it would be useful to have a method by which the correct number of parameter repetitions could be determined before the uncertainty dynamics are obtained. In that way conservatism due to excessive repetitions could be eliminated.

In the above example, the solutions of equations (71a) clearly show that the third appearance of \mathbf{a}' does not affect the system response. Recall, however, that the solutions are not unique. Not all solutions so readily show that \mathbf{a}' is excessively repeated. For example, if the “inspired guesses” in equations (74) are chosen so that

$$\mathbf{L}_{yv}(s) = \frac{1}{s^2 + \mathbf{d}s + \mathbf{e}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \quad (78)$$

then one solution for $\mathbf{L}_{eu}(s)$ is,

$$\mathbf{L}_{eu}^T(s) = \begin{bmatrix} k_1 & 0 & 1 - k_1 & 1 & 0 \\ k_2 & s & 1 - k_2 & 0 & 1 \end{bmatrix} \quad (79)$$

where k_1 and k_2 are arbitrary polynomials in s .

If $k_1 = k_2 = 1$, the third appearance of the perturbation \mathbf{a}' is not excited by any system input and, therefore, has no effect on the system response. Similarly, if $k_1 = k_2 = 0$, the first appearance of \mathbf{a}' has no effect on the system response. For other values of k_1 and k_2 , the fact that \mathbf{a}' only needs to appear twice is not obvious since each appearance of \mathbf{a}' is excited by the system input $\mathbf{u}(s)$ and generates a perturbation to the system output $\mathbf{d}(s)$. Therefore, it is not appropriate to use zero rows or columns alone to identify excessively repeated perturbations.

At present there is no reliable test using properties of the $\mathbf{L}_{yv}(s)$ and $\mathbf{L}_{eu}(s)$ matrices to identify excessively repeated perturbation parameters. The recommended approach is to start with a single independent parameter and sequentially add repeated parameters until the system of equations (e.g., eqs. (71)) can be solved, subject to the restrictions that the solutions be minimal polynomials.

5.3 Denominator Uncertainty for a Multivariable System

The approach to solving the denominator uncertainty problem is identical to the approach used for numerator uncertainties. The differences occur in the specific equations that need to be solved. The next example addresses generic denominator uncertainties to highlight these differences and to address the properties of the resulting system representation.

The system in equation (60) is considered once again, but in this case each of the denominator coefficients are uncertain and known to lie in sectors. That is,

$$\mathbf{d} = \underline{\mathbf{d}} + \mathbf{d}' \quad \mathbf{e} = \underline{\mathbf{e}} + \mathbf{e}' \quad (80)$$

where $\underline{\mathbf{d}}$ and $\underline{\mathbf{e}}$ are the nominal values of \mathbf{d} and \mathbf{e} and \mathbf{d}' and \mathbf{e}' are the perturbations of \mathbf{d} and \mathbf{e} . The sector bounds on the coefficients are such that the perturbation is centered at the origin with radius r (e.g., \mathbf{d}' is in a sector centered at zero with radius $r_{\mathbf{d}}$).

In this case the system frequency-domain input-output equation can be written as

$$(s^2 + \underline{\mathbf{d}}s + \underline{\mathbf{e}})\mathbf{y}(s) = \begin{bmatrix} s + \mathbf{a} & \mathbf{a} \\ s + \mathbf{b} & s^2 + \mathbf{a}s + \mathbf{c} \end{bmatrix} \mathbf{u}(s) - \begin{bmatrix} \mathbf{d}'s + \mathbf{e}' & 0 \\ 0 & \mathbf{d}'s + \mathbf{e}' \end{bmatrix} \mathbf{y}(s) \quad (81)$$

The perturbation response is, therefore,

$$\mathbf{d}(s) \triangleq -\frac{\mathbf{\Gamma}(s)}{\delta(s)} = \frac{-1}{s^2 + \underline{\mathbf{d}}s + \underline{\mathbf{e}}} \begin{bmatrix} \mathbf{d}'s + \mathbf{e}' & 0 \\ 0 & \mathbf{d}'s + \mathbf{e}' \end{bmatrix} \mathbf{y}(s) \quad (82)$$

where $\delta(s)$ is the denominator polynomial and $\mathbf{\Gamma}(s)$ is the matrix of numerator polynomials. From the block diagram in figure 4, equation (23a), and equation (82), one finds that two equations must be satisfied by the choice of uncertainty dynamics matrices. These equations are

$$-\frac{\mathbf{\Gamma}(s) \mathbf{N}_{yu}(s)}{\delta(s) d_{yu}(s)} = \frac{\mathbf{N}_{yv}(s)}{d_{yv}(s)} \delta \mathbf{C} \frac{\mathbf{N}_{eu}(s)}{d_{eu}(s)} \quad (83a)$$

$$-\frac{\mathbf{\Gamma}(s) \mathbf{N}_{yv}(s)}{\delta(s) d_{yv}(s)} = \frac{\mathbf{N}_{yv}(s)}{d_{yv}(s)} \delta \mathbf{C} \frac{\mathbf{N}_{ev}(s)}{d_{ev}(s)} \quad (83b)$$

where

$$\mathbf{L}_{yu}(s) = \frac{\mathbf{N}_{yu}(s)}{d_{yu}(s)} \quad \mathbf{L}_{yv}(s) = \frac{\mathbf{N}_{yv}(s)}{d_{yv}(s)} \quad (84a)$$

$$\mathbf{L}_{eu}(s) = \frac{\mathbf{N}_{eu}(s)}{d_{eu}(s)} \quad \mathbf{L}_{ev}(s) = \frac{\mathbf{N}_{ev}(s)}{d_{ev}(s)} \quad (84b)$$

and where the $\mathbf{N}(s)$ terms represent matrices of numerator polynomials and the $d(s)$ terms represent denominator polynomials.

Choose $d_{yv}(s) = \delta(s) = d_{yu}(s)$, consistent with the previous example, and choose $d_{eu}(s) = d_{ev}(s) = \delta(s)$. In this case equations (83) simplify to

$$-\mathbf{\Gamma}(s) \mathbf{N}_{yu}(s) = \mathbf{N}_{yv}(s) \delta \mathbf{C} \mathbf{N}_{eu}(s) \quad (85a)$$

$$-\mathbf{\Gamma}(s) \mathbf{N}_{yv}(s) = \mathbf{N}_{yv}(s) \delta \mathbf{C} \mathbf{N}_{ev}(s) \quad (85b)$$

As in the previous example, the solutions of equations (85) are not unique and, in order to make the solutions more tractable, the same restrictions are placed on the elements of the matrices $\mathbf{N}_{yu}(s)$, $\mathbf{N}_{yv}(s)$, $\mathbf{N}_{eu}(s)$, and $\mathbf{N}_{ev}(s)$. Thus, the elements of $\mathbf{N}(s)$ are required to be minimal polynomials (i.e., they only have terms involving either nonnegative integer powers of s or zero and are of the lowest possible order and the least possible number of terms).

The first step in solving equations (85) is to choose $\delta \mathbf{C}$ and its associated sector radius matrix \mathbf{R} . In this example, each uncertain parameter in $\delta \mathbf{C}$ must be repeated once, as if it is associated with two independent sectors. The repeated perturbation parameters \mathbf{d}' and \mathbf{e}' are required to characterize the overall effects of the uncertainty. Therefore, for this example,

$$\delta \mathbf{C} = \text{diag}[\mathbf{d}', \mathbf{d}', \mathbf{e}', \mathbf{e}'] \quad \mathbf{R} = \text{diag}[r_{\mathbf{d}}, r_{\mathbf{d}}, r_{\mathbf{e}}, r_{\mathbf{e}}] \quad (86)$$

One should expect the resulting sector bound to have some conservatism associated with the repeated perturbation parameters.

Next consider the uncertainty dynamics numerator matrices. For dimensional consistency $\mathbf{N}_{yv}(s)$ must be a 2×4 matrix, $\mathbf{N}_{eu}(s)$ must be a 4×2 matrix, and $\mathbf{N}_{ev}(s)$ must be a 4×4 matrix. Represent the numerator matrices as

$$\mathbf{N}_{yu}(s) = \begin{bmatrix} s + \mathbf{a} & \mathbf{a} \\ s + \mathbf{b} & s^2 + \mathbf{a}s + \mathbf{c} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \quad (87a)$$

$$\mathbf{\Gamma}(s) = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \quad \mathbf{N}_{yv}(s) = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \end{bmatrix} \quad (87b)$$

$$\mathbf{N}_{eu}^T(s) = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \end{bmatrix} \quad \mathbf{N}_{ev}(s) = \begin{bmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{bmatrix} \quad (87c)$$

Substituting equations (86) and (87) into equations (85), expanding the elements, and grouping like terms results in a system of equations for each uncertain parameter. For this example, the two systems that result can be written in an indicial form as follows.

System for \mathbf{d} :

$$\left. \begin{aligned} sk_{ij} &= l_{i1}m_{1j} + l_{i2}m_{2j} & (i, j = 1, 2) \\ sl_{ij} &= l_{i1}n_{1j} + l_{i2}n_{2j} & (i = 1, 2; j = 1, 2, 3, 4) \end{aligned} \right\} \quad (88)$$

System for \mathbf{e} :

$$\left. \begin{aligned} k_{ij} &= l_{i3}m_{3j} + l_{i4}m_{4j} & (i, j = 1, 2) \\ l_{ij} &= l_{i3}n_{3j} + l_{i4}n_{4j} & (i = 1, 2; j = 1, 2, 3, 4) \end{aligned} \right\} \quad (89)$$

Again, inspired guesses are used to solve these two systems of nonlinear equations. Let

$$l_{11} = l_{22} = 1 \quad l_{12} = l_{21} = 0 \quad l_{13} = l_{24} = 1 \quad l_{14} = l_{23} = 0 \quad (90)$$

The remaining numerator elements are determined next, subject to the imposed limitations discussed above. As a result the complete uncertainty dynamics representation becomes

$$\delta\mathbf{C} = \text{diag}[\mathbf{d}', \mathbf{d}', \mathbf{e}', \mathbf{e}'] \quad (91a)$$

$$\mathbf{L}_{eu}^T(s) = \frac{1}{s^2 + \underline{\mathbf{d}}s + \underline{\mathbf{e}}} \begin{bmatrix} s(s + \mathbf{a}) & s(s + \mathbf{b}) & s + \mathbf{a} & s + \mathbf{b} \\ s\mathbf{a} & s(s^2 + \mathbf{a}s + \mathbf{c}) & \mathbf{a} & s^2 + \mathbf{a}s + \mathbf{c} \end{bmatrix} \quad (91b)$$

$$\mathbf{L}_{yv}(s) = \frac{1}{s^2 + \underline{\mathbf{d}}s + \underline{\mathbf{e}}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad (91c)$$

$$\mathbf{L}_{ev}(s) = \frac{1}{s^2 + \underline{\mathbf{d}}s + \underline{\mathbf{e}}} \begin{bmatrix} s & 0 & s & 0 \\ 0 & s & 0 & s \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad (91d)$$

with the associated sector radius matrix \mathbf{R} from equations (86).

An example of the sector bound for the denominator uncertainty case (eq. (58)) is shown in figure 10. The same set of nominal values for the system parameters used in the previous example are used here as well; however, in this case the uncertain parameters are \mathbf{d} and \mathbf{e} . The nominal values and the radii for the uncertain parameter sectors are given by

$$\begin{aligned} \underline{\mathbf{a}} &= 5.4 & \underline{\mathbf{b}} &= 10.0 & \underline{\mathbf{c}} &= 25.25 & \underline{\mathbf{d}} &= 10.0 \\ \underline{\mathbf{e}} &= 100.0 & \mathbf{r}_d &= 5.0 & \mathbf{r}_e &= 40.0 \end{aligned}$$

Notice the conservatism associated with the need to treat the two uncertain parameters as four independent sectors when compared with the “exact” bound (based on real-valued perturbations). The exact bound is computed with constrained optimization, as in the previous example.

It is also interesting to note that one of the sufficiency conditions for existence of the sector bound (namely, $\bar{\sigma}[\mathbf{R}\mathbf{L}_{ev}(s)] < 1$) is violated for values of \mathbf{r}_e greater than 50. In order to compute a sector bound for $\mathbf{r}_e > 50$, one could scale the inputs and outputs of the uncertainty dynamics $\delta\mathbf{L}_{yu}(s)$ to reduce the norm of \mathbf{R} .

Figure 10. Sector-based uncertainty bound for denominator uncertainty.

5.4 Multiple Parameter Uncertainties in a Second-Order System

The solution of the sector bound for a general system is accomplished by combining the solution methods for numerator and denominator uncertainties. A general representation of the uncertainty disturbance $\mathbf{d}(s)$ can be represented as a function of the inputs and outputs of the true system:

$$\mathbf{d}(s) = \mathbf{\Gamma}_1(s) \mathbf{u}(s) - \mathbf{\Gamma}_2(s) \mathbf{y}(s) \tag{92}$$

The transfer matrices $\mathbf{\Gamma}_i$ relate the inputs and outputs of the true system to the uncertainty disturbance. The uncertainty disturbance expression in equation (92) can be combined with equation (23a) to obtain a relation in terms of $\mathbf{u}(s)$ and $\mathbf{v}(s)$:

$$\mathbf{d}(s) = [\mathbf{\Gamma}_1(s) - \mathbf{\Gamma}_2(s) \mathbf{L}_{yu}(s)] \mathbf{u}(s) - \mathbf{\Gamma}_2(s) \mathbf{L}_{yv}(s) \mathbf{v}(s) \tag{93}$$

The uncertainty disturbance can be written in an alternate form by inspection of the block diagram depicted in figure 4:

$$\mathbf{d}(s) = \mathbf{L}_{yv}(s) \delta \mathbf{C} [\mathbf{L}_{eu}(s) \mathbf{u}(s) + \mathbf{L}_{ev}(s) \mathbf{v}(s)] \quad (94)$$

Equating the two expressions for the uncertainty disturbance (eqs. (93) and (94)) and grouping the \mathbf{u} -terms and \mathbf{v} -terms results in two expressions that the uncertainty dynamics must satisfy:

$$\mathbf{\Gamma}_1(s) - \mathbf{\Gamma}_2(s) \mathbf{L}_{yu}(s) = \mathbf{L}_{yv}(s) \delta \mathbf{C} \mathbf{L}_{eu}(s) \quad (95a)$$

$$-\mathbf{\Gamma}_2(s) \mathbf{L}_{yv}(s) = \mathbf{L}_{yv}(s) \delta \mathbf{C} \mathbf{L}_{ev}(s) \quad (95b)$$

Note that the two expressions in equations (95) are algebraic in nature. They correspond to a set of scalar equations that are nonlinear in the unknown transfer function elements. The solutions of these equations are similar to those of the special cases discussed previously. The solutions are not unique because the equations underspecify the solution. Analogous to the previous examples, however, a solution is readily obtained by imposing the same practical restrictions on allowable forms of the solution.

In order to see how the generalized solution approach can be applied to a problem with physical significance, consider the transfer function in equation (96). The nominal system is unity but has unmodeled dynamics that are known to be in the form of a second-order damped oscillator.⁴ This system has a true transfer function of the form

$$\frac{\mathbf{y}(s)}{\mathbf{u}(s)} = \mathbf{G}(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2} = \mathbf{G}_0(s) + \Delta(s) = 1 - \frac{s^2 + 2\zeta\omega s}{s^2 + 2\zeta\omega s + \omega^2} \quad (96)$$

where the frequency ω and damping ratio ζ are uncertain but are bounded as follows:

$$0.4 \leq \zeta \leq 0.8$$

$$22 \leq \omega \leq 27 \text{ rad/sec}$$

A sector bound for this example is computed two ways. The first way might be termed the naive approach, in which each appearance of an uncertain parameter is treated as an independent uncertainty. The second solution approach makes use of the physical structure of the system to recognize that the same parameter appears in both the numerator and denominator of the system transfer function. One would expect the second approach to give a less conservative representation of the uncertainty, and this will be shown to be the case.

For the first case the system is modeled as shown in equation (97):

$$\mathbf{G}_1(s) = \frac{\mathbf{a}}{s^2 + \mathbf{b}s + \mathbf{c}} \quad (97)$$

The parameters \mathbf{a} , \mathbf{b} , and \mathbf{c} are independent and related to ω and ζ in such a way that the nominal parameter values and sector radii are

$$\begin{array}{lll} \underline{\mathbf{a}} = 606.5 & \underline{\mathbf{b}} = 30.4 & \underline{\mathbf{c}} = 606.5 \\ \mathbf{r}_a = 122.5 & \mathbf{r}_b = 12.8 & \mathbf{r}_c = 122.5 \end{array}$$

That is, \mathbf{a} and \mathbf{c} lie in sectors centered at 606.5 with radii 122.5, and \mathbf{b} lies in a sector centered at 30.4 with radius 12.8.

⁴ This example is adapted from the example in Stein and Doyle (1978).

The perturbation matrix $\delta\mathbf{C}$ and the $\mathbf{\Gamma}$ matrices are given by

$$\delta\mathbf{C} = \begin{bmatrix} \mathbf{a}' & 0 & 0 \\ 0 & \mathbf{b}' & 0 \\ 0 & 0 & \mathbf{c}' \end{bmatrix} \quad \mathbf{\Gamma}_1(s) = \frac{\mathbf{a}'}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{c}}} \quad \mathbf{\Gamma}_2(s) = \frac{\mathbf{b}'s + \mathbf{c}'}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{c}}} \quad (98)$$

where the diagonal structure of the $\delta\mathbf{C}$ matrix is dictated by sector theory. One solution of equations (95) for these values of $\delta\mathbf{C}$, $\mathbf{\Gamma}_1$, and $\mathbf{\Gamma}_2$, with $\mathbf{G}_1(s)$ substituted for $\mathbf{L}_{yu}(s)$, is summarized in equations (99):

$$\mathbf{L}_{yv}(s) = \frac{1}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{c}}} [1, 1, 1] \quad (99a)$$

$$\mathbf{L}_{eu}^T(s) = \frac{1}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{c}}} [s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{c}}, -\underline{\mathbf{a}}s, -\underline{\mathbf{a}}] \quad (99b)$$

$$\mathbf{L}_{ev}(s) = \frac{1}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{c}}} \begin{bmatrix} 0 & 0 & 0 \\ -s & -s & -s \\ -1 & -1 & -1 \end{bmatrix} \quad (99c)$$

The sector radius matrix for this case is

$$\mathbf{R} = \begin{bmatrix} r_a & 0 & 0 \\ 0 & r_b & 0 \\ 0 & 0 & r_c \end{bmatrix} \quad (100)$$

Notice that $\mathbf{L}_{yv}(s)$, $\mathbf{L}_{ev}(s)$, and $\mathbf{L}_{eu}(s)$ have the same denominator corresponding to the characteristic polynomial of the nominal plant. This is not required but significantly simplifies the solution and the structure of the uncertainty dynamics. This particular solution, once the denominators are chosen, is determined by the choice of the elements of the numerator of the $\mathbf{L}_{yv}(s)$ matrix. The remaining numerator terms are then specified by the governing relations, equations (95).

Also notice that the upper row of the $\mathbf{L}_{ev}(s)$ matrix, the one associated with the uncertain numerator parameter \mathbf{a} , is zero. This is because the $\mathbf{L}_{ev}(s)$ matrix is the feedback block inside the uncertainty dynamics block diagram (fig. 4) and is required only when there is uncertainty in the denominator parameters. Since \mathbf{a} only appears in the numerator of $\mathbf{G}_1(s)$ no feedback around the $\delta\mathbf{C}$ matrix is needed.

The sector-based uncertainty bound corresponding to equation (54) for the above solution (three independent parameters) is presented in figure 11. The ‘‘absolute’’ uncertainty bound determined by constrained optimization methods is also presented for comparison. Notice the significant conservatism associated with the sector-based bound, especially at low frequencies. This conservatism can be attributed to several factors, as discussed previously. The source of most of the conservatism, however, is the treatment of the parameters as three independent uncertainties. This is demonstrated in the next example.

Figure 11. Comparison of uncertainty bounds.

Consider the system modeled as shown in equation (101), in which the parameter \mathbf{a} appears in both the numerator and denominator. This accounts for the fact that in the actual system, equation (96), this is indeed the case. The nominal parameter values and sector radii for \mathbf{a} and \mathbf{b} are the same as for the previous case.

$$\mathbf{G}_2(s) = \frac{\mathbf{a}}{s^2 + \mathbf{b}s + \mathbf{a}} \quad (101)$$

The perturbation matrix $\delta\mathbf{C}$ and the $\mathbf{\Gamma}$ matrices for this case are given by

$$\delta\mathbf{C} = \begin{bmatrix} \mathbf{a}' & 0 \\ 0 & \mathbf{b}' \end{bmatrix} \quad \mathbf{\Gamma}_1(s) = \frac{\mathbf{a}'}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{a}}} \quad \mathbf{\Gamma}_2(s) = \frac{\mathbf{b}'s + \mathbf{a}'}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{a}}} \quad (102)$$

A solution for the uncertainty dynamics for the two-parameter case, shown in equations (103), is very similar to the three-parameter solution, with the exception that equations (103) are of lower order:

$$\mathbf{L}_{yv}(s) = \frac{1}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{a}}} [1, 1] \quad (103a)$$

$$\mathbf{L}_{eu}^T(s) = \frac{1}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{a}}} [s^2 + \underline{\mathbf{b}}s, -\underline{\mathbf{a}}s] \quad (103b)$$

$$\mathbf{L}_{ev}(s) = \frac{1}{s^2 + \underline{\mathbf{b}}s + \underline{\mathbf{a}}} \begin{bmatrix} -1 & -1 \\ -s & -s \end{bmatrix} \quad (103c)$$

The sector radius matrix for this case is also of lower order:

$$\mathbf{R} = \begin{bmatrix} r_{\mathbf{a}} & 0 \\ 0 & r_{\mathbf{b}} \end{bmatrix} \quad (104)$$

The sector-based uncertainty bound corresponding to equation (58) for the case of two independent uncertain parameters is presented in figure 11 along with the “absolute” uncertainty bound and the three-parameter bound. Notice that this bound is considerably less conservative than that of the previous case. This is attributable to the fact that the physical structure of the true system is more accurately represented. Some conservatism remains, however, because of the reasons cited previously. Conservatism is also introduced because the parameters \mathbf{a} and \mathbf{b} , while treated as such, are not truly independent of one another, since both depend on the value of ω .

The examples presented above were used to demonstrate the manner in which conservatism enters the sector-based uncertainty bound. In addition, the examples led the development of a general form for the equations that govern the uncertainty dynamics $\delta\mathbf{L}_{yu}$ (eqs. (95)). The nonuniqueness of the solution to the governing equations is one source of conservatism that was demonstrated in the first example. The need to repeat single uncertain parameters multiple times is another source of conservatism that was demonstrated in the second example. The need to treat multiple dependent parameters as independent is yet another source of conservatism that was demonstrated in the third example. The need to repeat some parameters and treat dependent parameters as independent can in most instances be traced to the requirement that the perturbation matrix $\delta\mathbf{C}$ be diagonal.

The examples have also shown that the sector-based uncertainty bound provides a means to readily compute, in closed form, a frequency-dependent characterization of the influence of individual uncertain parameters on the overall frequency response of an LTI-MIMO system. This characterization of the system uncertainty can be directly applied to robustness tests and robust control design tools.

6 Concluding Remarks

This paper has addressed several aspects of applying sector stability theory to the problem of characterizing plant model uncertainty. Sector properties were applied to linear multivariable systems with Safonov’s approach. A general matrix representation for linear, time-invariant, multivariable systems was presented to facilitate the application of the sector properties. The resulting sector-based uncertainty bound was combined with a representative singular-value robustness condition to obtain a useful robustness measure that accounts for known uncertainty in the plant model. Finally, the sector-based approach was applied to several forms of parameter uncertainty to highlight the specific issues involved in applying the sector-based approach to linear, time-invariant, multivariable systems and to identify properties of the sector bound associated with fundamental forms of plant model uncertainty.

Based on the results obtained, the characteristic advantages and limitations of the sector-based approach are clarified. Three primary advantages of the sector-based approach become evident: the approach provides a way to effectively describe overall plant model uncertainty when bounds on individual model parameters are known, the approach is multivariable by nature and so directly applies to linear multivariable dynamical systems, and the approach is compatible with established singular-value-based robustness measures.

The major limitations associated with this approach are the conservatism of the resulting uncertainty bound and the difficulty in solving the systems of nonlinear equations that determine the uncertainty dynamics. Conservatism may be introduced by the diagonal matrix representation of the uncertain parameters required to apply sector concepts to the uncertainty modeling problem. Each uncertain parameter is assumed to be independent of all other uncertain parameters. Conservatism is introduced whenever repeated parameters are required. Mathematical conservatism is introduced by the process of merging the sector bound with singular-value concepts. This conservatism can be attributed to two main factors: the merging process requires

the use of inequalities, and the uncertainties are assumed to be complex valued with an arbitrary phase angle (i.e., bounded only in magnitude). Another factor that may have some effect on the degree to which the sector bound is conservative is associated with the nonuniqueness of the uncertainty dynamics representation. The systems of nonlinear equations associated with the uncertain parameters can be readily solved, as shown in the examples, but they are underspecified. The level of conservatism is increased if the solution of the nonlinear equations results in excessively repeated uncertain parameters.

A feature of the methodology that has not yet been exploited effectively is the nonuniqueness of the solution to the uncertainty dynamics equations. There may be a “best” solution that minimizes the overall effect of the various forms of conservatism on the accuracy of the sector-based uncertainty bound. The additional degrees of freedom might be able to be used to generate such a solution, which results in the least conservative sector bound. Solution methods that exploit the sector approach and result in the least conservative uncertainty bound are subjects for future research.

NASA Langley Research Center
Hampton, VA 23665-5225
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Appendix A

Proofs

Proof of Lemma A1

Lemma A1. Given two complex matrices \mathbf{A} and \mathbf{B} such that \mathbf{A} is $\underline{n} \times \underline{n}$, $\mathbf{A}^* \equiv \mathbf{A}$, and $\mathbf{A} > 0$ (i.e., positive definite), \mathbf{B} is $r \times n$, $\text{rank}(\mathbf{B}) = r$, and $r \leq n$,

$$\tilde{\mathbf{A}} \triangleq \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \quad (\text{A1})$$

then,

$$\mathbf{A} \geq \mathbf{B}^* \tilde{\mathbf{A}} \mathbf{B} \quad (\text{A2})$$

(i.e., $\mathbf{x}^* \mathbf{A} \mathbf{x} \geq \mathbf{x}^* \mathbf{B}^* \tilde{\mathbf{A}} \mathbf{B} \mathbf{x}$).

Proof. Define

$$\mathbf{B}^\dagger \triangleq \mathbf{A}^{-1} \mathbf{B}^* \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \quad (\text{A3})$$

which is the right pseudoinverse of \mathbf{B} with respect to \mathbf{A} . Then,

$$\begin{aligned} \tilde{\mathbf{A}} &\triangleq \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} = \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \\ &= \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \mathbf{B} \mathbf{A}^{-1} \mathbf{A} \mathbf{A}^{-1} \mathbf{B}^* \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \\ &= \left[\mathbf{B}^\dagger \right]^* \mathbf{A} \mathbf{B}^\dagger \end{aligned} \quad (\text{A4})$$

Now consider

$$\begin{aligned} \mathbf{x}^* \left[\mathbf{A} - \mathbf{B}^* \tilde{\mathbf{A}} \mathbf{B} \right] \mathbf{x} &= \mathbf{x}^* \mathbf{A} \mathbf{x} - \mathbf{x}^* \mathbf{B}^* \tilde{\mathbf{A}} \mathbf{B} \mathbf{x} \\ &= \mathbf{x}^* \mathbf{A} \mathbf{x} - \mathbf{x}^* \mathbf{B}^* \left[\mathbf{B}^\dagger \right]^* \mathbf{A} \mathbf{B}^\dagger \mathbf{B} \mathbf{x} \\ &= \mathbf{x}^* \mathbf{A} \mathbf{x} - \mathbf{x}^* \left[\mathbf{B}^\dagger \mathbf{B} \right]^* \mathbf{A} \left[\mathbf{B}^\dagger \mathbf{B} \right] \mathbf{x} \end{aligned} \quad (\text{A5})$$

where \mathbf{x} is an appropriately dimensioned complex vector. In addition, since $\mathbf{B} \mathbf{B}^\dagger = \mathbf{I}$ (by definition) and since

$$\begin{aligned} \left[\mathbf{B}^\dagger \mathbf{B} \right] \left[\mathbf{B}^\dagger \mathbf{B} \right] &= \mathbf{A}^{-1} \mathbf{B}^* \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \mathbf{B} \\ &= \mathbf{A}^{-1} \mathbf{B}^* \left[\mathbf{B} \mathbf{A}^{-1} \mathbf{B}^* \right]^{-1} \mathbf{B} \\ &= \mathbf{B}^\dagger \mathbf{B} \end{aligned} \quad (\text{A6})$$

then $\mathbf{B}^\dagger \mathbf{B}$ is a projection such that

$$\mathbf{B}^\dagger \mathbf{B} \mathbf{x} = \mathbf{x}_{\eta^\perp(\mathbf{B})} \quad (\text{A7})$$

where $\mathbf{x}_{\eta^\perp(\mathbf{B})}$ is a vector that lies in the orthogonal complement of the null space of \mathbf{B} . Both the null space of \mathbf{B} and its orthogonal complement are spaces of lower dimension than those of \mathbf{A} .

Any vector \mathbf{x} can be written as the sum of two components, one that lies in the null space of $\mathbf{B}, \eta(\mathbf{B})$, and one that lies in its orthogonal complement, $\eta^\perp(\mathbf{B})$, so that

$$\mathbf{x} = \mathbf{x}_{\eta(\mathbf{B})} + \mathbf{x}_{\eta^\perp(\mathbf{B})} \quad (\text{A8})$$

As a result,

$$\begin{aligned} \mathbf{x}^* (\mathbf{A} - \mathbf{B}^* \tilde{\mathbf{A}} \mathbf{B}) \mathbf{x} &= \mathbf{x}^* \mathbf{A} \mathbf{x} - \mathbf{x}_{\eta^\perp(\mathbf{B})}^* \mathbf{A} \mathbf{x}_{\eta^\perp(\mathbf{B})} \\ &= \mathbf{x}_{\eta(\mathbf{B})}^* \mathbf{A} \mathbf{x}_{\eta(\mathbf{B})} \end{aligned} \quad (\text{A9})$$

Since \mathbf{A} is given to be positive definite and since $\mathbf{x}_{\eta(\mathbf{B})}$ lies in a space of lower dimension than \mathbf{A} , then

$$\mathbf{A} - \mathbf{B}^* \tilde{\mathbf{A}} \mathbf{B} \geq 0 \quad (\text{A10})$$

(positive semidefinite) and so

$$\mathbf{A} \geq \mathbf{B}^* \tilde{\mathbf{A}} \mathbf{B} \quad (\text{A11})$$

Sector Bound Existence Tests

In this section simple tests to check for the existence of a system sector bound are developed and proven. In order to calculate a system sector bound several matrices need to be formed. Some of these matrices require the inverses of other matrices to exist. It is advisable to check for the existence of these matrix inverses before attempting to determine the sector bound itself. Two tests are developed to determine the existence of these matrix inverses.

Existence test for \mathbf{Q}_v^{-1} . The inverse of \mathbf{Q}_v must exist in order to calculate the matrices in equations (40). For a matrix to have an inverse it must be definite. Since \mathbf{Q}_v is Hermitian it must be positive definite. This fact is used to develop the existence test.

Since \mathbf{Q}_v must be positive definite,

$$\mathbf{x}^* \mathbf{Q}_v \mathbf{x} > 0 \quad (\text{A12})$$

$$\Leftrightarrow \mathbf{x}^* [\mathbf{I} - \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev}] \mathbf{x} > 0 \quad (\text{A13})$$

$$\Leftrightarrow \mathbf{x}^* \mathbf{I} \mathbf{x} - \mathbf{x}^* \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev} \mathbf{x} > 0 \quad (\text{A14})$$

Now if $\mathbf{x}^* \mathbf{x} \neq 0$, then

$$\Leftrightarrow \frac{\mathbf{x}^* \mathbf{L}_{ev}^* \mathbf{R}^* \mathbf{R} \mathbf{L}_{ev} \mathbf{x}}{\mathbf{x}^* \mathbf{x}} < 1 \quad (\text{A15})$$

Using the Euclidean norm notation gives

$$\Leftrightarrow \frac{\|\mathbf{R} \mathbf{L}_{ev} \mathbf{x}\|_E^2}{\|\mathbf{x}\|_E^2} < 1 \quad (\text{A16})$$

which is guaranteed to be true if the following singular-value inequality is satisfied:

$$\Leftrightarrow \bar{\sigma}(\mathbf{R} \mathbf{L}_{ev}) < 1 \quad (\text{A17})$$

Therefore, equation (A17) is a simple test for the existence of \mathbf{Q}_v^{-1} .

Existence test for \mathbf{Q}_n . Since \mathbf{Q}_n is defined to be the inverse of a matrix product, that inverse must exist. Recall that

$$\mathbf{Q}_n \triangleq \left(\mathbf{L}_{yv} \mathbf{Q}_v^{-1} \mathbf{L}_{yv}^* \right)^{-1} \quad (\text{A18})$$

For \mathbf{Q}_n to exist, \mathbf{Q}_v^{-1} must exist and $\mathbf{L}_{yv} \mathbf{Q}_v^{-1} \mathbf{L}_{yv}^*$ must be positive definite. Assume that the existence of \mathbf{Q}_v^{-1} has already been established.

Since \mathbf{Q}_v^{-1} exists, it must be full rank and it can be factored (e.g., by Cholesky decomposition). Let the factors be defined by

$$\mathbf{Q}_v^{-1} \triangleq \mathbf{F} \mathbf{F}^* \quad (\text{A19})$$

Therefore, it must be shown that

$$\mathbf{L}_{yv} \mathbf{F} \mathbf{F}^* \mathbf{L}_{yv}^* > 0 \quad (\text{A20})$$

Note that \mathbf{Q}_v is square and let its dimension (and rank) be n . Then the dimension (and rank) of \mathbf{F} is also n . Note also that \mathbf{L}_{yv} is not necessarily square; let its dimensions be $m \times n$. Clearly then, $\mathbf{L}_{yv} \mathbf{F} \mathbf{F}^* \mathbf{L}_{yv}^*$ is an $m \times m$ matrix. Such a matrix is only positive definite if (1) $m \leq n$ and (2) $\text{rank}(\mathbf{L}_{yv}) = m$.

If $m > n$, then $\mathbf{L}_{yv} \mathbf{F} \mathbf{F}^* \mathbf{L}_{yv}^*$ is deficient because the rank of \mathbf{L}_{yv} can be at most n and so the rank of $\mathbf{L}_{yv} \mathbf{F} \mathbf{F}^* \mathbf{L}_{yv}^*$ can be at most n as well. Also, since the matrix product $\mathbf{L}_{yv} \mathbf{F} \mathbf{F}^* \mathbf{L}_{yv}^*$ is Hermitian it will be positive definite if it is full rank. Therefore, if condition (1) is satisfied and the rank of \mathbf{L}_{yv} is m , then $\mathbf{L}_{yv} \mathbf{F} \mathbf{F}^* \mathbf{L}_{yv}^*$ is of full rank.

Both of the tests presented above can be readily performed numerically. They should be performed as part of the algorithm used to determine system sector bounds to ensure that the problem is properly formulated and that the bound resulting from such an algorithm is valid.

Appendix B

Sector Bound Algorithm

The sector algorithm is coded with command files for MATRIX χ ,⁵ a commercially available linear systems analysis and design package. Three separate command files are used: *safsys.dat* is used to set up the uncertainty dynamics matrices, *safbnd.dat* is used to set up the sector bound matrix, and *calcs.dat* is used to actually compute the sector-based uncertainty bound. An additional command file, *logdist.dat*, is used to create a frequency vector with points evenly distributed on a logarithmic scale.

safsys.dat

The command file *safsys.dat* is problem specific and needs to be customized for each solution of the uncertainty dynamics matrices (i.e., \mathbf{L}_{yu} , \mathbf{L}_{yv} , \mathbf{L}_{ev} , and \mathbf{L}_{eu}). A listing of *safsys.dat* for the denominator uncertainty example from section 5.3 is presented below.

```
//  
//          Command File - safsys.dat  
//          =====  
//  
// Author   : Martin R. Waszak  
// Date     : 6/29/90  
//  
// Description : Formulate the representation of the system for use  
//               in calculating Safonov's sector uncertainty bound.  
//  
//           The formulation is problem dependent and must be reprogrammed  
//           for every new problem to be addressed.  
//  
//           This formulation only applies for the denominator uncertainty  
//           example case which appears in section 5.3.  
//  
a = 5.4; b = 10.0; c = 25.25; d = 10.0; e = 100.0;  
svar = jay*omega(i);  
den = svar**2 + d*svar + e;  
ns = 2;nv = 4;  
g0 = 1/den*[svar+a, a; svar+b, svar**2+a*svar+c];  
gyu = 1/den*[svar+a, a; svar+b, svar**2+a*svar+c];  
geu = 1/den*[ svar*(svar+a), svar*(svar+b), svar+a, svar+b;...  
             svar*a, svar*(svar**2+a*svar+c), a, svar**2+a*svar+c];  
gyv = 1/den*[ 1 0 1 0; 0 1 0 1 ];  
gev = 1/den*[ svar, 0, svar, 0 ; 0, svar, 0, svar;1, 0, 1, 0; 0, 1, 0, 1];  
  
clear svar den a b d c e  
return
```

⁵ MATRIX χ is a registered trademark of IntegratedSystems Inc.

safbnd.dat

The command file *safbnd.dat* is problem specific and needs to be customized for each choice of the perturbation matrix $\delta\mathbf{C}$. The sector radius matrix is determined from the individual parameter sectors associated with the uncertain parameters represented in $\delta\mathbf{C}$. A listing of *safbnd.dat* for the denominator uncertainty example from section 5.3 is presented below.

```
//
//          Command File - safbnd.dat
//          =====
//
// Author   : Martin R. Waszak
// Date     : 6/29/90
//
// Description : Formulate the sector radius matrix associated
//               with the sector bounds on each of the uncertain elements.
//
//          This formulation only applies for the denominator uncertainty
//          example case in section 5.3.
//
r = diag([5.0,5.0,40.0,40.0]);
return
```

calcs.dat

The command file *calcs.dat* computes the sector-based uncertainty bound described in equation (58). It applies for all forms of uncertainty dynamics (e.g., additive, multiplicative reflected at the plant input, and multiplicative reflected at the plant output). A listing of *calcs.dat* is presented below.

```
//
//          Command File - calcs.dat
//          =====
//
// Author   : Martin R. Waszak
// Date     : 6/29/90
//
// Description : Calculate a singular value bound on model
//               uncertainty based on sector stability theory.
//
// Remarks   : This algorithm can determine sector bounds, for both
//               modeled parameter uncertainties and for unmodeled dynamics
//               with known structure but with uncertain parameters, where
//               the uncertainties can be represented by sectors.
//
//          The variable names do not correspond to the matrices
//          defined in the theoretical sector formulations.
//          They do have some relationship with the 'standard'
//          notation, however. They have been altered
//          to improve the computational aspects of the algorithm.
//
```

```

// Inputs      : logdist.cmd   - command file to formulate frequency vector
//                                     evenly spaced on a log scale
//              omega         - vector of frequencies (from logdist.cmd)
//              n              - number of frequencies
//
//              safsys.dat     - command file to formulate
//                                     system representation
//              g0, gyu, gyv,
//              geu, gev       - system matrices (from safsys.dat)
//              ns             - size of the nominal system (from safsys.dat)
//              nv             - number of component uncertainties
//                                     (from safsys.dat)
//
//              safbnd.dat     - command file to formulate sector
//                                     component uncertainty bound matrix
//              r              - matrix of sector bounds
//                                     (from safbnd.dat)
//
// Outputs     : bs           - system sector bound
//
// Modified    : 7/7/88       - to correct the expressions for 't1' and 't2'
//              : 4/18/89     - to modify the sector bound equation; bs(i)
//              : 7/12/89     - to calculate svd(r*gev) for existence test
//              : 6/29/90     - to apply for additive uncertainties
//
//=====
//
// Obtain frequency vector evenly spaced on a log scale
//
wmin=0.1;wmax=100;npts=300;exec('logdist.cmd')
n=300;
//
// Perform the bound calculation for each frequency (FOR LOOP)
//
display('Forming system matrices and bounds - (this will take a while)'),...
for i=1:n;...
...//
...// Obtain the special sector system representation
...// and matrix of element sector bounds
exec('safbnd'); exec('safsys');...
...//
...// Calculate the necessary matrices
t1 = r*gev; t2=r*geu;...
...// Calculate svd(r*gev) which should be < 1 for existence
sv1=svd(t1);...
if sv1(1) >= 1.0 then display('existence test violated'), return, end,...
m1 = inv( eye(nv) - t1'*t1 ); m2 = inv( eye(nv) - t1*t1' );...
q =real( gyv*m1*gyv' ); p = real( t2'* m2*t2 );...
gc = gyu + gyv*m1*t1'*t2;...
...//
...// Calculate Cholesky factors

```

```

cfq = chol(q); cfp = chol(p);...
...//
...// Calculate singular values of appropriate matrices
sv1 = svd(gc - g0);...
sv2 = svd(cfq); sv3 = svd(cfp);...
...//
...// Calculate sector bound and then loop back for another frequency
bs(i) = ( sv1(1) + sv2(1)*sv3(1) ) ; end
//
display('All done! - sector bound stored in "bs" ')
//
clear g0 gc gyu gyv geu gev ns nv m1 m2 t1 t2 n p q r
clear cfq cfp sv1 sv2 sv3 sv4
return

```

logdist.dat

The command file *logdist.dat* is used to generate a logarithmically distributed frequency vector in a range between specified minimum and maximum values.

```

//
//          Command File - logdist.cmd
//          =====
//
// Author   : Martin R. Waszak
// Date     : 4/12/88
//
// Description : This function calculates a range of points that
//               are uniformly distributed on a log10 scale. It is ideally
//               suited for use in plotting things on a log scale
//               (for example - bode plots, singular value plots, etc.).
//
// Inputs    : The following temporary variables must be defined
//               in your matrix-x stack. They will be cleared from the stack
//               after the vector of points is formed.
//
//           npts   = # of data points
//           wmin   = minimum value of the range of values
//           wmax   = maximum value of the range of values
//
// Outputs   : The range of logarithmically distributed
//               values will be output in the following vector -
//
//           omega  = vector of range of values
//
// Remarks   : Since this is a command file, the variables used
//               here are global (not local) and care must be taken that
//               these variable names are unassigned in your calling
//               program.
//

```

```

// Modified      : 6/29/88 - to display notification that calculation has
//                                     begun
//               : 6/30/88 - to improve documentation and improve
//                                     generality
//
//=====
//
display('Calculating a vector of points evenly spaced on a log scale.')
//
for i=1:npts; omega(i) = wmax**(i/npts) * wmin**((npts-i)/npts);
//
display('The logarithmically distributed range of -')
npts
display('values with minimum value -')
wmin
display('and maximum value -')
wmax
display('is stored in the column vector "omega." ')
display('Note : wmin, wmax and npts have been cleared from the stack.')
clear wmin wmax npts
return

```

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Nomenclature

A	arbitrary matrix (complex or real valued depending on application)
B	arbitrary matrix (complex or real valued depending on application)
C (<i>s</i>)	sector center matrix
c	sector center; uncertain parameter
d	uncertain parameter
d (<i>s</i>)	perturbation response
$d_{eu}(s)$	denominator polynomial of $\mathbf{L}_{eu}(s)$
$d_{ev}(s)$	denominator polynomial of $\mathbf{L}_{ev}(s)$
$d_{yv}(s)$	denominator polynomial of $\mathbf{L}_{yv}(s)$
e	uncertain parameter
e (<i>s</i>)	input to perturbation matrix $\delta\mathbf{C}$
F (<i>s</i>)	matrix of complex-valued functions
$f(\bullet)$	function of (\bullet)
G (<i>s</i>)	nominal plant transfer matrix
K (<i>s</i>)	feedback compensation
k_i	arbitrary polynomial
k_{ij}	i, j element of $\mathbf{N}_{yu}(s)$
$\mathbf{L}_{eu}(s)$	transfer matrix that transforms the vector of system inputs
$\mathbf{L}_{ev}(s)$	transfer matrix that allows parameters in $\delta\mathbf{C}$ to appear in denominator of the uncertainty dynamics
$\mathbf{L}_{yu}(s)$	transfer matrix for the nominal plant model
$\mathbf{L}_{yv}(s)$	transfer matrix that transforms the output of the perturbation block into the perturbation response
l_{ij}	i, j element of $\mathbf{N}_{yv}(s)$
m_{ij}	i, j element of $\mathbf{N}_{eu}(s)$
$\mathbf{N}_{eu}(s)$	numerator of $\mathbf{L}_{eu}(s)$
$\mathbf{N}_{yv}(s)$	numerator of $\mathbf{L}_{yv}(s)$
n_{ij}	i, j element of $\mathbf{N}_{ev}(s)$
P_n	defined in equation (44b)
P_v	defined in equation (34)
Q_n	defined in equation (44c)
Q_v	defined in equation (28a)
R (<i>s</i>)	sector radius matrix
R_v	defined in equation (28b)
r	sector radius

$r(\bullet)$	sector radius for parameter (\bullet)
\mathbf{S}_v	defined in equation (28c)
s	Laplace variable
$\mathbf{T}(s)$	linear, time-invariant, multivariable operator
\mathbf{T}_n	defined in equation (44a)
\mathbf{T}_v	defined in equation (30)
t	independent variable
$\mathbf{u}(s)$	plant input vector
$u(t)$	arbitrary real function of t
$\mathbf{v}(s)$	output from perturbation matrix $\delta\mathbf{C}$
\mathbf{x}	arbitrary vector compatible with \mathbf{A}
$\mathbf{y}(s)$	plant output vector
Δ	uncertainty representation
$\tilde{\Delta}$	second term on right side of equation (44a)
$\delta(s)$	uncertainty dynamics denominator polynomial
$\delta\mathbf{C}$	perturbation matrix
δc	perturbation from sector center
$\delta\mathbf{L}_{yu}$	uncertainty dynamics
$\mathbf{\Gamma}_{(\bullet)}(s)$	matrices of uncertainty dynamics numerator polynomial
$\bar{\sigma}(\bullet)$	maximum singular value of (\bullet)
$\underline{\sigma}(\bullet)$	minimum singular value of (\bullet)
ω	frequency
ζ	uncertain parameter; damping ratio
$(\hat{\bullet})$	Fourier transform of (\bullet)
$(\underline{\bullet})$	nominal value of (\bullet)
$(\bullet)'$	perturbation of (\bullet)
$(\bullet)^*$	complex conjugate transpose of (\bullet)
$ \bullet $	absolute value of (\bullet)
$\ \bullet\ _2$	2-norm
$\ \bullet\ _E$	Euclidean norm of (\bullet)
$\langle(\bullet), (\bullet)\rangle$	inner product

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